

MULTI-DIMENSIONAL STOCHASTIC SINGULAR CONTROL VIA DYNKIN GAME AND DIRICHLET FORM

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Abstract. The traditional difficulty about stochastic singular control is to characterize the existence and regularities of the value function and the optimal control policy. In this paper, a multi-dimensional singular control problem is considered. We found the optimal value function and the optimal control policy of this problem via Dynkin game, whose solution is given by the saddle point of the cost function. The existence and uniqueness of the solution to this Dynkin game are proved through an associated variational inequality problem involving Dirichlet form. As a consequence, the properties of the value function of this Dynkin game implies the smoothness of the value function of the stochastic singular control problem. In this way, we are able to show the existence of a classical solution to this multi-dimensional singular control problem, which was traditionally solved in the sense of viscosity solutions.¹

Key words. Dynkin game, Dirichlet form, Multi-dimensional diffusion, Stochastic singular control

AMS subject classifications. 49J40, 60G40, 60H30, 91A80, 93E20

1. Introduction and Problem Formulation. The one dimensional optimal stopping problem has been thoroughly studied, see, e.g., Dayanik and Karatzas [3], Gapeev and Lerche [8]. Given a symmetric Markov process on a locally compact separable metric space, it is well known that the solution of an optimal stopping problem admits its quasi continuous version of the solution to a variational inequality problem involving Dirichlet form, e.g., see Nagai [14]. Zabczyk [17] extended this result to a zero-sum stopping game, also known as Dynkin game. In the one dimensional case, the integrated form of the value function of the Dynkin game was identified to be the solution of an associated stochastic singular control problem, e.g., see Taksar [15], Fukushima and Taksar [6]. As a result, the classical smooth solution can be obtained for this singular control problem other than the viscosity solution techniques [4] [2] [1]. The approach to solve singular control problems through variational inequalities and Dynkin game can also be found in Karatzas and Zamfirescu [12], Guo and Tomecek [9].

In the one dimensional singular control problem, each point in the space has a positive capacity ([6]), hence the nonexistence of the proper exceptional set. However, this is no longer the case in multi-dimensional singular control problem. This paper is an extension of the work by Fukushima and Taksar [6] to multi-dimensional stochastic singular control problem. We overcome this difficulty using the absolute continuity of the transition function of the underlying process [7].

In this paper, we are concerned with a multi-dimensional diffusion on \mathbb{R}^n :

$$(1.1) \quad d\mathbf{X}_t = \mu(\mathbf{X}_t)dt + \sigma(\mathbf{X}_t)d\mathbf{B}_t,$$

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where

$$\mathbf{X}_t = \begin{pmatrix} X_{1t} \\ \vdots \\ X_{nt} \end{pmatrix}, \mu(\mathbf{X}_t) = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_n \end{pmatrix}, \sigma(\mathbf{X}_t) = \begin{pmatrix} \sigma_{11} & \cdots & \sigma_{1m} \\ \vdots & & \vdots \\ \sigma_{n1} & \cdots & \sigma_{nm} \end{pmatrix}, \mathbf{B}_t = \begin{pmatrix} B_{1t} \\ \vdots \\ B_{mt} \end{pmatrix}, \quad (1.2)$$

in which $\mu_i, \sigma_{i,j}$ ($1 \leq i \leq n, 1 \leq j \leq m$) are functions of $X_{1t}, \dots, X_{(n-1)t}$, and \mathbf{B}_t is m -dimensional Brownian motion with $m \geq n$. Thus we are given a system $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbf{X}, \theta_t, P_{\mathbf{x}})$, where (Ω, \mathcal{F}) is a measurable space, $\mathbf{X} = \mathbf{X}(\omega)$ is a mapping of Ω into $C(\mathbb{R}^n)$, $\mathcal{F}_t = \sigma(\mathbf{X}_s, s \leq t)$, and θ_t is a shift operator in Ω such that $\mathbf{X}_s(\theta_t \omega) = \mathbf{X}_{s+t}(\omega)$. Here $P_{\mathbf{x}}(\mathbf{x} \in \mathbb{R}^n)$ is a family of measures under which $\{\mathbf{X}_t, t \geq 0\}$ is an n -dimensional diffusion with initial state \mathbf{x} .

We assume the following usual conditions:

ASSUMPTION 1.1. *There exist constants C and D such that*

$$|\mu(\mathbf{x})| + |\sigma(\mathbf{x})| \leq C(1 + |\mathbf{x}|), \quad \mathbf{x} \in \mathbb{R}^n;$$

$$|\mu(\mathbf{x}) - \mu(\mathbf{y})| + |\sigma(\mathbf{x}) - \sigma(\mathbf{y})| \leq D|\mathbf{x} - \mathbf{y}|, \quad \mathbf{x}, \mathbf{y} \in \mathbb{R}^n.$$

A control policy is defined as a pair $(A_t^{(1)}, A_t^{(2)}) = \mathcal{S}$ of \mathcal{F}_t adapted processes which are right continuous and nondecreasing in t and we assume $A_0^{(1)}, A_0^{(2)}$ are nonnegative. Denote \mathbb{S} the set of all admissible policies, whose detailed definition will be given in Section 4.

Given a policy $\mathcal{S} = (A_t^{(1)}, A_t^{(2)}) \in \mathbb{S}$ we define the following controlled process:

$$\begin{aligned} dX_{1t} &= \mu_1 dt + \sigma_{11} dB_{1t} + \cdots + \sigma_{1m} dB_{mt}, \\ &\vdots \\ dX_{nt} &= \mu_n dt + \sigma_{n1} dB_{1t} + \cdots + \sigma_{nm} dB_{mt} + dA_t^{(1)} - dA_t^{(2)}, \\ \mathbf{X}_0 &= \mathbf{x}, \end{aligned}$$

with the cost function

$$(1.3) k_{\mathcal{S}}(\mathbf{x}) = E_{\mathbf{x}} \left(\int_0^\infty e^{-\alpha t} h(\mathbf{X}_t) dt + \int_0^\infty e^{-\alpha t} \left(f_1(\mathbf{X}_t) dA_t^{(1)} + f_2(\mathbf{X}_t) dA_t^{(2)} \right) \right), \quad (1.4)$$

$$f_1(\mathbf{x}), f_2(\mathbf{x}) > 0, \quad \forall \mathbf{x} \in \mathbb{R}^n.$$

Here we assume $A_t^{(1)} - A_t^{(2)}$ is the minimal decomposition of a bounded variation process into a difference of two increasing processes.

There are two types of costs associated with the process \mathbf{X}_t for each policy \mathcal{S} . The first one is the holding cost $h(\mathbf{X}_t)$ accumulated along time. The second one is the control cost associated with the processes $(A_t^{(1)}, A_t^{(2)})$, and this cost increases only when $(A_t^{(1)}, A_t^{(2)})$ increase.

One looks for a control policy \mathcal{S} that minimizes $k_{\mathcal{S}}(\mathbf{x})$, i.e.,

$$(1.5) \quad W(\mathbf{x}) = \min_{\mathcal{S} \in \mathbb{S}} k_{\mathcal{S}}(\mathbf{x}).$$

As an application of this model, a decision maker observes the expenses of a company under a multi-factor situation but only has control over one factor, yet she still wants to minimize the total expected cost. Analogously, by studying the

associated maximization problem, i.e., taking the negative of min, this model can be used to find the optimal investment policy where an investor observes the prices of several assets in a portfolio but could only adjust one of them.

The rest of this paper is organized as follows: we first introduce some preliminaries on Dirichlet form and a variational inequality problem in Section 2. In Section 3 we identify conditions for the value function as well as the optimal policy of the associated Dynkin game. The integral form of the value function of this Dynkin game is shown in Section 4 to be the value of a multi-dimensional singular control problem, and the optimal control policy is also determined consequently.

2. Dirichlet Form and a Variational Inequality Problem. Let \mathbb{X} be a locally compact separable metric space, m be an everywhere dense positive Radon measure on \mathbb{X} , and $L^2(\mathbb{X}, m)$ denotes the L^2 space on \mathbb{X} . We assume that the Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathbb{X}, m)$ is regular in the sense that $\mathcal{F} \cap C_0(\mathbb{X})$ is \mathcal{E}_1 dense in \mathcal{F} and is uniformly dense in $C_0(\mathbb{X})$, where the \mathcal{E}_1 norm is defined as follows:

$$\|u\|_{\mathcal{E}_1} = \left(\mathcal{E}(u, u) + \int_{\mathbb{X}} u(\mathbf{x})^2 m(d\mathbf{x}) \right)^{1/2}.$$

Analogously we define $\mathcal{E}_\alpha(u, v)$ as $\mathcal{E}_\alpha(u, v) = \mathcal{E}(u, v) + \alpha(u, v)$ ($\alpha > 0$), where

$$(u, v) = \int_{\mathbb{X}} u(\mathbf{x})v(\mathbf{x})m(d\mathbf{x}).$$

In this case $(\mathcal{E}, \mathcal{F})$ is called regular [5].

For this Dirichlet form, there exists an associated Hunt process $\mathbf{M} = (\mathbf{X}_t, P_{\mathbf{x}})$ on \mathbb{X} , see [5], such that

$$p_t f(\mathbf{x}) := E_{\mathbf{x}} f(\mathbf{X}_t), \quad \mathbf{x} \in \mathbb{X}$$

is a version of $T_t f$ for all $f \in C_0(\mathbb{X})$, where T_t is the L^2 semigroup associated with the Dirichlet form $(\mathcal{E}, \mathcal{F})$. Furthermore, the L^2 -resolvent $\{G_\alpha, \alpha > 0\}$ associated with this Dirichlet form satisfies

$$(2.1) \quad G_\alpha f \in \mathcal{F}, \quad \mathcal{E}_\alpha(G_\alpha f, u) = (f, u), \quad \forall f \in L^2(\mathbb{X}; m), \quad \forall u \in \mathcal{F},$$

and the resolvent $\{R_\alpha, \alpha > 0\}$ of the Hunt process \mathbf{M} given by

$$R_\alpha f(\mathbf{x}) = E_{\mathbf{x}} \left(\int_0^\infty e^{-\alpha t} f(\mathbf{X}_t) dt \right), \quad \mathbf{x} \in \mathbb{X},$$

is a quasi-continuous modification of $G_\alpha f$ for any Borel function $f \in L^2(\mathbb{X}; m)$.

For $\alpha > 0$, a measurable function f on \mathbb{X} is called α -excessive if $f(\mathbf{x}) \geq 0$ and $e^{-\alpha t} p_t f(\mathbf{x}) \uparrow f(\mathbf{x})$ as $t \downarrow 0$ for any $\mathbf{x} \in \mathbb{X}$. A function $f \in \mathcal{F}$ is said to be an α -potential if $\mathcal{E}_\alpha(f, g) \geq 0$ for any $g \in \mathcal{F}$ with $g \geq 0$. For any α -potential $f \in \mathcal{F}$, define $\hat{f}(\mathbf{x}) = \lim_{t \downarrow 0} p_t f(\mathbf{x})$, then $f = \hat{f}$ m -a.e. and \hat{f} is α -excessive (see Section 3 in [7]). \hat{f} is called the α -excessive regularization of f . Furthermore, any α -excessive function is finely continuous (see Theorem A.2.7 in [5]).

Let $f_1, f_2 \in \mathcal{F}$ be finely continuous functions such that for all $\mathbf{x} \in \mathbb{X}$

$$(2.2) \quad -f_1(\mathbf{x}) \leq f_2(\mathbf{x}), \quad |f_1(\mathbf{x})| \leq \phi(\mathbf{x}), \quad |f_2(\mathbf{x})| \leq \psi(\mathbf{x}),$$

where ϕ, ψ are some finite α -excessive functions. We further define the set

$$(2.3) \quad K = \{u \in \mathcal{F} : -f_1 \leq u \leq f_2, \text{ m-a.e.}\}.$$

For a given continuous and bounded function $H \in L^2(\mathbb{X}; m)$ one looks for a solution $V \in K$ to the following variational inequality problem

$$(2.4) \quad \mathcal{E}_\alpha(V, u - V) \geq (H, u - V), \quad \forall u \in K.$$

Here we can see $R_\alpha H$ (hence $G_\alpha H$) is also a bounded function.

As related literature, Nagai [14] considered an optimal stopping problem and showed that there exist a quasi continuous function $w \in \mathcal{F}$ which solves the variational inequality

$$w \geq -f_1, \quad \mathcal{E}_\alpha(w, u - w) \geq 0, \quad \forall u \in \mathcal{F} \text{ with } u \geq -f_1,$$

and a properly exceptional set \mathbb{N} such that for all $\mathbf{x} \in \mathbb{X}/\mathbb{N}$,

$$w(\mathbf{x}) = \sup_{\sigma} E_{\mathbf{x}} \left(e^{-\alpha\sigma} [-f_1(\mathbf{X}_\sigma)] \right) = E_{\mathbf{x}} \left(e^{-\alpha\hat{\sigma}} [-f_1(\mathbf{X}_{\hat{\sigma}})] \right),$$

where

$$\hat{\sigma} = \inf\{t \geq 0; w(\mathbf{X}_t) = -f_1(\mathbf{X}_t)\}.$$

Moreover, w is the smallest α -potential dominating the function $-f_1$ m-a.e.

Zabczyk [17] then extended this result to the solution of the zero-sum game-also called Dynkin game-by showing that, there exist a quasi continuous function $V(\mathbf{x}) \in K$ which solves the variational inequality

$$(2.5) \quad \mathcal{E}_\alpha(V, u - V) \geq 0, \quad \forall u \in K,$$

and a properly exceptional set \mathbb{N} such that for all $\mathbf{x} \in \mathbb{X}/\mathbb{N}$,

$$(2.6) \quad V(\mathbf{x}) = \sup_{\sigma} \inf_{\tau} J_{\mathbf{x}}(\tau, \sigma) = \inf_{\tau} \sup_{\sigma} J_{\mathbf{x}}(\tau, \sigma)$$

for any stopping times τ and σ , where

$$(2.7) \quad J_{\mathbf{x}}(\tau, \sigma) = E_{\mathbf{x}} \left(e^{-\alpha(\tau \wedge \sigma)} (-I_{\sigma \leq \tau} f_1(\mathbf{X}_\sigma) + I_{\tau < \sigma} f_2(\mathbf{X}_\tau)) \right).$$

Let

$$E_1 = \{\mathbf{x} \in \mathbb{X}/\mathbb{N} : V(\mathbf{x}) = -f_1(\mathbf{x})\}, \quad E_2 = \{\mathbf{x} \in \mathbb{X}/\mathbb{N} : V(\mathbf{x}) = f_2(\mathbf{x})\},$$

then the hitting times $\hat{\tau} = \tau_{E_2}$, $\hat{\sigma} = \tau_{E_1}$ is the saddle point of the game

$$(2.8) \quad J_{\mathbf{x}}(\hat{\tau}, \sigma) \leq J_{\mathbf{x}}(\hat{\tau}, \hat{\sigma}) \leq J_{\mathbf{x}}(\tau, \hat{\sigma})$$

for any $\mathbf{x} \in \mathbb{X}/\mathbb{N}$ and any stopping times τ, σ , where $J_{\mathbf{x}}$ is given in (2.7). In particular, the value of the Dynkin game is given by

$$(2.9) \quad V(\mathbf{x}) = J_{\mathbf{x}}(\hat{\tau}, \hat{\sigma}), \quad \forall \mathbf{x} \in \mathbb{X}/\mathbb{N}.$$

Fukushima and Menda [7] showed that, if the transition function of \mathbf{M} satisfies an absolute continuity condition, i.e.,

$$(2.10) \quad p_t(\mathbf{x}, \cdot) \ll m(\cdot),$$

for all $t > 0$ and $\mathbf{x} \in \mathbb{X}$, and f_1, f_2 satisfy the following separability condition:
There exist finite α -excessive functions $v_1, v_2 \in \mathcal{F}$ such that, for all $\mathbf{x} \in \mathbb{X}$,

$$(2.11) \quad -f_1(\mathbf{x}) \leq v_1(\mathbf{x}) - v_2(\mathbf{x}) \leq f_2(\mathbf{x}),$$

then Zabczyk's result still holds and there does not exist the exceptional set \mathbf{N} . The following is a version of Theorem 2 in [7].

THEOREM 2.1. *Assume conditions (2.2), (2.10) and (2.11). There exists a finite finely continuous function V satisfying the variational inequality (2.5) and the identity*

$$V(\mathbf{x}) = \sup_{\sigma} \inf_{\tau} J_{\mathbf{x}}(\tau, \sigma) = \inf_{\tau} \sup_{\sigma} J_{\mathbf{x}}(\tau, \sigma), \quad \forall \mathbf{x} \in \mathbb{X},$$

where σ, τ range over all stopping times. Moreover, the pair $(\hat{\tau}, \hat{\sigma})$ defined by

$$\hat{\tau} = \inf\{t > 0 : V(\mathbf{X}_t) = f_2(\mathbf{X}_t)\}, \quad \hat{\sigma} = \inf\{t > 0 : V(\mathbf{X}_t) = -f_1(\mathbf{X}_t)\},$$

is the saddle point of the game in the sense that

$$J_{\mathbf{x}}(\hat{\tau}, \sigma) \leq J_{\mathbf{x}}(\hat{\tau}, \hat{\sigma}) \leq J_{\mathbf{x}}(\tau, \hat{\sigma}), \quad \forall \mathbf{x} \in \mathbb{X},$$

for all stopping times τ, σ .

In the case $H \neq 0$ in (2.4), we have the following proposition:

PROPOSITION 2.2. *There exists a unique finite finely continuous function $V \in K$ which solves (2.4).*

Proof. For the case $H = 0$, by Theorem 2.1 there exists a finite finely continuous function $V \in K$ such that

$$(2.12) \quad \mathcal{E}_{\alpha}(V, u - V) \geq 0, \quad \forall u \in K.$$

From (2.12) we have

$$\mathcal{E}_{\alpha}(V, u) \geq \mathcal{E}_{\alpha}(V, V) \geq 0.$$

By Cauchy-Schwarz inequality,

$$\mathcal{E}_{\alpha}(u, u)\mathcal{E}_{\alpha}(V, V) \geq \mathcal{E}_{\alpha}^2(V, u),$$

and this implies

$$(2.13) \quad \mathcal{E}_{\alpha}(u, u) \geq \mathcal{E}_{\alpha}(V, V).$$

Conversely, suppose (2.13) holds, and define $w = u - V$ for an arbitrary $u \in K$. Since K is convex, it can be seen that

$$V + \epsilon w = (1 - \epsilon)V + \epsilon u \in K, \quad \forall \epsilon \in (0, 1).$$

Thus (2.13) gives

$$\mathcal{E}_{\alpha}(V, V) \leq \mathcal{E}_{\alpha}(V + \epsilon w, V + \epsilon w),$$

or equivalently

$$0 \leq 2\mathcal{E}_\alpha(V, w) + \epsilon\mathcal{E}_\alpha(w, w).$$

By letting $\epsilon \rightarrow 0$ we get (2.12).

By virtue of the closedness and convexity of the set K , we conclude that (2.13) (or equivalently (2.12)) has a unique solution.

Now for a nonzero H in (2.4), because H is continuous, (2.1) implies that $G_\alpha H$ is a continuous and bounded function. Define $h_1 = f_1 + G_\alpha H$, $h_2 = f_2 - G_\alpha H$, then by the conditions on H in (2.4) and the conditions on f_1, f_2 in (2.2), it can be seen that h_1, h_2 are finite finely continuous functions. Let

$$(2.14) \quad K^0 = \{u \in \mathcal{F}; -h_1 \leq u \leq h_2, \text{ m-a.e.}\},$$

and $V^0 = V - G_\alpha H$, then it can be concluded that

$$(2.15) \quad \mathcal{E}_\alpha(V^0, u - V^0) \geq 0, \quad \forall u \in K^0$$

has a unique solution. (2.15) implies

$$\mathcal{E}_\alpha(V - G_\alpha H, u - G_\alpha H - (V - G_\alpha H)) \geq 0, \quad \forall u \in K,$$

which can be reduced to (2.4) by virtue of (2.1). \square

We assume the following separability condition:

ASSUMPTION 2.1. *There exist finite α -excessive functions $v_1, v_2 \in \mathcal{F}$ such that, for all $\mathbf{x} \in \mathbb{X}$,*

$$(2.16) \quad -f_1(\mathbf{x}) - G_\alpha H(\mathbf{x}) \leq v_1(\mathbf{x}) - v_2(\mathbf{x}) \leq f_2(\mathbf{x}) - G_\alpha H(\mathbf{x}),$$

then the following results holds:

THEOREM 2.3. *For any bounded and continuous function H given in (2.4) and for any $f_1, f_2 \in \mathcal{F}$ satisfying (2.2) (2.16), we put*

$$(2.17) \quad J_\mathbf{x}(\tau, \sigma) = E_\mathbf{x} \left(\int_0^{\tau \wedge \sigma} e^{-\alpha t} H(\mathbf{X}_t) dt \right)$$

$$(2.18) \quad + E_\mathbf{x} \left(e^{-\alpha(\tau \wedge \sigma)} (-I_{\sigma \leq \tau} f_1(\mathbf{X}_\sigma) + I_{\tau < \sigma} f_2(\mathbf{X}_\tau)) \right)$$

for any stopping times τ, σ . Then the solution of (2.3) and (2.4) admits a finite finely continuous value function of the game

$$(2.19) \quad V(\mathbf{x}) = \inf_\tau \sup_\sigma J_\mathbf{x}(\tau, \sigma) = \sup_\sigma \inf_\tau J_\mathbf{x}(\tau, \sigma), \quad \forall \mathbf{x} \in \mathbb{X}.$$

Furthermore if we let

$$E_1 = \{\mathbf{x} \in \mathbb{X} : V(\mathbf{x}) = -f_1(\mathbf{x})\}, \quad E_2 = \{\mathbf{x} \in \mathbb{X} : V(\mathbf{x}) = f_2(\mathbf{x})\},$$

then the hitting times $\hat{\tau} = \tau_{E_2}$, $\hat{\sigma} = \tau_{E_1}$ is the saddle point of the game

$$(2.20) \quad J_\mathbf{x}(\hat{\tau}, \sigma) \leq J_\mathbf{x}(\hat{\tau}, \hat{\sigma}) \leq J_\mathbf{x}(\tau, \hat{\sigma})$$

for any $\mathbf{x} \in \mathbb{X}$ and any stopping times τ, σ . In particular,

$$(2.21) \quad V(\mathbf{x}) = J_\mathbf{x}(\hat{\tau}, \hat{\sigma}), \quad \forall \mathbf{x} \in \mathbb{X}.$$

Proof. We sketch the proof given in [6] which used the result in [17]. The case where $H = 0$ is due to Theorem 2.1. For a nonzero H satisfying (2.4), (2.15) admits a finite finely continuous solution

$$V^0 = \inf_{\tau} \sup_{\sigma} J_{\mathbf{x}}^0(\tau, \sigma) = \sup_{\sigma} \inf_{\tau} J_{\mathbf{x}}^0(\tau, \sigma), \quad \forall \mathbf{x} \in \mathbb{X},$$

and

$$J_{\mathbf{x}}^0(\tau, \sigma) = E_{\mathbf{x}} \left(e^{-\alpha(\tau \wedge \sigma)} (-I_{\sigma \leq \tau} h_1(\mathbf{X}_{\sigma}) + I_{\tau < \sigma} h_2(\mathbf{X}_{\tau})) \right),$$

where $h_1 = -f_1 + R_{\alpha}H$, $h_2 = f_2 - R_{\alpha}H$ are finite finely continuous functions.

Applying Dynkin formula to $e^{-\alpha t} R_{\alpha}H(\mathbf{X}_t)$ yields

$$\begin{aligned} & E_{\mathbf{x}} \left(e^{-\alpha(\tau \wedge \sigma)} R_{\alpha}H(\mathbf{X}_{\tau \wedge \sigma}) \right) \\ &= R_{\alpha}H(\mathbf{x}) + E_{\mathbf{x}} \left(\int_0^{\tau \wedge \sigma} \mathcal{L} e^{-\alpha t} R_{\alpha}H(\mathbf{X}_t) dt \right) \\ &= R_{\alpha}H(\mathbf{x}) + E_{\mathbf{x}} \left(\int_0^{\tau \wedge \sigma} (-\alpha e^{-\alpha t} R_{\alpha}H(\mathbf{X}_t) + e^{-\alpha t} \mathcal{L} R_{\alpha}H(\mathbf{X}_t)) dt \right) \\ &= R_{\alpha}H(\mathbf{x}) - E_{\mathbf{x}} \left(\int_0^{\tau \wedge \sigma} e^{-\alpha t} H(\mathbf{X}_t) dt \right), \end{aligned}$$

where \mathcal{L} is the infinitesimal generator, and this leads to

$$J_{\mathbf{x}}(\tau, \sigma) = J_{\mathbf{x}}^0(\tau, \sigma) + R_{\alpha}H(\mathbf{x}).$$

Now it can be seen that the variational inequality problem (2.3) and (2.4) admits a finite finely continuous solution

$$V(\mathbf{x}) = V^0(\mathbf{x}) + R_{\alpha}H(\mathbf{x}), \quad \mathbf{x} \in \mathbb{X}.$$

□

3. The Dynkin Game and Its Value Function. Two players P_1 and P_2 observe a multi-dimensional underlying process \mathbf{X}_t in (1.1) with accumulated income, discounted at present time, equalling $\int_0^{\sigma} e^{-\alpha t} H(\mathbf{X}_t) dt$ for any stopping time σ . If P_1 stops the game at time σ , he pays P_2 the amount of the accumulated income plus the amount $f_2(\mathbf{X}_{\sigma})$, which after been discounted equals $e^{-\alpha \sigma} f_2(\mathbf{X}_{\sigma})$. If the process is stopped by P_2 at time σ , he receives from P_1 the accumulated income less the amount $f_1(\mathbf{X}_{\sigma})$, which after been discounted equals $e^{-\alpha \sigma} f_1(\mathbf{X}_{\sigma})$. P_1 tries to minimize his payment while P_2 tries to maximize his income. The value of this game is thus given by

$$(3.1) \quad V(\mathbf{x}) = \inf_{\tau} \sup_{\sigma} J_{\mathbf{x}}(\tau, \sigma), \quad \forall \mathbf{x} \in \mathbb{R}^n,$$

where $J_{\mathbf{x}}$ is given by (2.17) on \mathbb{R}^n .

ASSUMPTION 3.1. $H(\mathbf{x}) = H(x_1, \dots, x_n)$ is a continuous function on \mathbb{R}^n such that

$$H(\bar{\mathbf{x}}, 0) = 0, \quad \forall \bar{\mathbf{x}} \in \mathbb{R}^{n-1},$$

and for any $\mathbf{x} \in \mathbb{R}^n$, $H(\mathbf{x})$ is strictly increasing in x_n . Furthermore, $H(\mathbf{x})$ satisfies a uniform Lipschitz condition on x_n : there exist $L_1, L_2 > 0$ such that

$$L_1|x_{n1} - x_{n2}| \leq |H(x_1, \dots, x_{n-1}, x_{n1}) - H(x_1, \dots, x_{n-1}, x_{n2})| \leq L_2|x_{n1} - x_{n2}|.$$

Define $A(\bar{\mathbf{x}}), B(\bar{\mathbf{x}})$ to be smooth and uniformly Lipschitz functions on \mathbb{R}^{n-1} such that $-C_1 < A(\bar{\mathbf{x}}) < C_2 < 0 < C_2 < B(\bar{\mathbf{x}}) < C_1$, $\forall \bar{\mathbf{x}} \in \mathbb{R}^{n-1}$ for constants $C_1, C_2 > 0$. The determination of $A(\bar{\mathbf{x}}), B(\bar{\mathbf{x}})$ will be given shortly. Define the region

$$\mathbb{D} = (-\infty, \infty) \times \dots \times (-\infty, \infty) \times (A, B) = \mathbb{R}^{n-1} \times (A, B), \quad \bar{\mathbf{x}} \in \mathbb{R}^{n-1},$$

which is a subset of \mathbb{R}^n . It can be seen that $H(\mathbf{x})$ is uniformly bounded in \mathbb{D} , so is $R_\alpha H(\mathbf{x})$, $\mathbf{x} \in \mathbb{D}$.

ASSUMPTION 3.2. $f_1(\mathbf{x}), f_2(\mathbf{x}) > 0 (\forall \mathbf{x} \in \mathbb{R}^n)$ are bounded and smooth functions such that $-M < -f_1(\mathbf{x}) < -\beta < 0 < \beta < f_2(\mathbf{x}) < M$, $\forall \mathbf{x} \in \mathbb{R}^n$ for fixed constants $M, \beta > 0$, and further

$$\frac{\partial f_1}{\partial x_n}(\mathbf{x}) > 0, \quad \frac{\partial f_2}{\partial x_n}(\mathbf{x}) < 0, \quad \forall \mathbf{x} \in \mathbb{D},$$

$$f_1(\bar{\mathbf{x}}, 0) = c_1, \quad f_2(\bar{\mathbf{x}}, 0) = c_2, \quad \forall \bar{\mathbf{x}} \in \mathbb{R}^{n-1} \quad \text{where } c_1 > 0, c_2 > 0 \text{ are constants.}$$

$$\mathcal{L}f_1(\bar{\mathbf{x}}, x_n) \text{ is strictly decreasing in } x_n \text{ for fixed } \bar{\mathbf{x}}, \quad \forall \mathbf{x} \in \mathbb{R}^{n-1} \times (A(\bar{\mathbf{x}}), 0),$$

$$\mathcal{L}f_2(\bar{\mathbf{x}}, x_n) \text{ is strictly increasing in } x_n \text{ for fixed } \bar{\mathbf{x}}, \quad \forall \mathbf{x} \in \mathbb{R}^{n-1} \times (0, B(\bar{\mathbf{x}})).$$

REMARK 3.1. Notice that $\beta/2$ and 0 are two α -excessive functions, and $-f_1(\mathbf{x}) < \beta/2 - 0 < f_2(\mathbf{x})$. If we define $H^+(\mathbf{x}) = \max(H(\mathbf{x}), 0)$, $H^-(\mathbf{x}) = \min(H(\mathbf{x}), 0)$, then $G_\alpha H^+(\mathbf{x})$ and $-G_\alpha H^-(\mathbf{x})$ are α -excessive functions. Put $v_1(\mathbf{x}) = \beta/2 - G_\alpha H^-(\mathbf{x})$ and $v_2(\mathbf{x}) = G_\alpha H^+(\mathbf{x})$, then the separability condition (2.16) holds for $-f_1$ and f_2 .

Consider again the diffusion (1.1). Define its infinitesimal generator \mathcal{L} as

$$(3.2) \quad \mathcal{L} := \sum_{i=1}^n \mu_i \frac{\partial}{\partial x_i} + \sum_{i,j=1}^n A_{ij} \frac{\partial^2}{\partial x_i \partial x_j},$$

where $[A_{ij}] = \mathbf{A}(\mathbf{x}) = \frac{1}{2}\sigma\sigma^T$. Notice that since σ_{ij} 's are functions of $X_{1t}, \dots, X_{(n-1)t}$, so is \mathbf{A} . For $\bar{\mathbf{x}} \in \mathbb{R}^{n-1}$, we assume $\mathbf{A}(\bar{\mathbf{x}})$ is positive definite and there exist two constants ϵ_1, ϵ_2 such that $0 < \epsilon_1 < \det(\mathbf{A}(\bar{\mathbf{x}})) < \epsilon_2 < \infty$ for any $\bar{\mathbf{x}} \in \mathbb{R}^{n-1}$. It can be seen that \mathcal{L} is uniformly elliptic. Define the measure $m(d\mathbf{x}) = e^{\mathbf{b} \cdot \mathbf{x}} d\mathbf{x}$, where $\mathbf{b} = \mathbf{A}^{-1}\mu$, then it is obvious that m is a positive Radon measure. For the generator \mathcal{L} , its associated Dirichlet form $(\mathcal{E}, \mathcal{F})$ densely embedded in $L^2(\mathbb{D}; m)$ is then given by

$$(3.3) \quad \mathcal{E}(u, v) = \int_{\mathbb{D}} \nabla u(\mathbf{x}) \cdot \mathbf{A} \nabla v(\mathbf{x}) m(d\mathbf{x}), \quad u, v \in \mathcal{F},$$

where

$$\mathcal{F} = \{u \in L^2(\mathbb{D}) : u \text{ is absolutely continuous, } \int_{\mathbb{D}} \nabla u(\mathbf{x})^T \nabla u(\mathbf{x}) m(d\mathbf{x}) < \infty\}.$$

Because f_1, f_2 are continuous and the underlying process possesses continuous sample paths, obviously f_1, f_2 are also finely continuous functions.

ASSUMPTION 3.3. For any stopping time σ we have

$$f_1(\mathbf{x}) \leq E_{\mathbf{x}}(e^{-\alpha\sigma} f_1(\mathbf{X}_\sigma)) + 4M(1 - E_{\mathbf{x}}(e^{-\alpha\sigma})),$$

$$f_2(\mathbf{x}) \leq E_{\mathbf{x}}(e^{-\alpha\sigma} f_2(\mathbf{X}_\sigma)) + 4M(1 - E_{\mathbf{x}}(e^{-\alpha\sigma})),$$

where M was given in Assumption 3.2.

ASSUMPTION 3.4. *There exist smooth and uniformly Lipschitz continuous functions $a(\bar{\mathbf{x}}), b(\bar{\mathbf{x}})$, $\bar{\mathbf{x}} \in \mathbb{R}^{n-1}$, such that,*

$$\begin{aligned} (\alpha - \mathcal{L})f_1(\bar{\mathbf{x}}, a(\bar{\mathbf{x}})) + H(\bar{\mathbf{x}}, a(\bar{\mathbf{x}})) &= 0, \\ (\alpha - \mathcal{L})f_2(\bar{\mathbf{x}}, b(\bar{\mathbf{x}})) - H(\bar{\mathbf{x}}, b(\bar{\mathbf{x}})) &= 0, \\ \text{and } A(\bar{\mathbf{x}}) < a(\bar{\mathbf{x}}) < 0 < b(\bar{\mathbf{x}}) < B(\bar{\mathbf{x}}), \quad \forall \bar{\mathbf{x}} \in \mathbb{R}^{n-1}. \end{aligned}$$

REMARK 3.2. *Because $a(\bar{\mathbf{x}}), b(\bar{\mathbf{x}})$ are uniformly Lipschitz and \mathbf{X}_t is uniformly elliptic, each boundary point of the region $\mathbb{R}^{n-1} \times (a, b)$ is a regular boundary point for the process \mathbf{X}_t . In fact, the uniform interior cone condition is satisfied, e.g., see Section 2 in [16].*

We are now concerned with a solution $V \in \mathcal{F}$, $V \geq -f_1$ of

$$(3.4) \quad \mathcal{E}_\alpha(V, u - V) \geq (H, u - V), \quad \forall u \in \mathcal{F}, \quad u \geq -f_1,$$

and by a similar proof of Proposition 2.2 and the fact that $G_\alpha H(\mathbf{x})$ is continuous by the dominated convergence theorem, it can be shown that there exists a unique finite and continuous solution V to (3.4). Again by the dominated convergence theorem and the absolute continuity of the transition probability function p_t , it can be easily shown that the point wise limit of V , given by

$$(3.5) \quad \hat{V}(\mathbf{x}) = \lim_{t \rightarrow 0} p_t V(\mathbf{x}), \quad \mathbf{x} \in \mathbb{D},$$

is finite and continuous. What is more, \hat{V} is the solution to the optimal stopping problem with holding cost H , which is a slight extension of Theorem 1 in [7]. In what follows we may still use V to denote \hat{V} since $V = \hat{V}$ m -a.e.

With the newly defined $(\mathcal{E}_\alpha, \mathcal{F})$, the conditions on H and f_1, f_2 , and together with the holding of the separability condition, we can, by a similar proof to Theorem 2 in [7], extend Theorem 2.3 as follows :

THEOREM 3.1. *For the functions H satisfying Assumption 3.1 and $f_1, f_2 \in \mathcal{F}$ satisfying Assumption 3.2, we put*

$$\begin{aligned} (3.6) \quad J_\mathbf{x}(\tau, \sigma) &= E_\mathbf{x} \left(\int_0^{\tau \wedge \sigma} e^{-\alpha t} H(\mathbf{X}_t) dt \right) \\ (3.7) \quad &+ E_\mathbf{x} \left(e^{-\alpha(\tau \wedge \sigma)} (-I_{\sigma \leq \tau} f_1(\mathbf{X}_\sigma) + I_{\tau < \sigma} f_2(\mathbf{X}_\tau)) \right) \end{aligned}$$

for any stopping times τ, σ . Then the solution of (2.3) and (2.4) admits a finite and continuous value function of the game

$$(3.8) \quad V(\mathbf{x}) = \inf_\tau \sup_\sigma J_\mathbf{x}(\tau, \sigma) = \sup_\sigma \inf_\tau J_\mathbf{x}(\tau, \sigma), \quad \forall \mathbf{x} \in \mathbb{R}^n.$$

Furthermore if we let

$$E_1 = \{\mathbf{x} \in \mathbb{R}^n : V(\mathbf{x}) = -f_1(\mathbf{x})\}, \quad E_2 = \{\mathbf{x} \in \mathbb{R}^n : V(\mathbf{x}) = f_2(\mathbf{x})\},$$

then the hitting times $\hat{\tau} = \tau_{E_2}$, $\hat{\sigma} = \tau_{E_1}$ is the saddle point of the game

$$(3.9) \quad J_\mathbf{x}(\hat{\tau}, \sigma) \leq J_\mathbf{x}(\hat{\tau}, \hat{\sigma}) \leq J_\mathbf{x}(\tau, \hat{\sigma})$$

for any $\mathbf{x} \in \mathbb{R}^n$ and any stopping times τ, σ . In particular,

$$(3.10) \quad V(\mathbf{x}) = J_{\mathbf{x}}(\hat{\tau}, \hat{\sigma}), \quad \forall \mathbf{x} \in \mathbb{R}^n.$$

In the rest of this section we will characterize the optimal control policies and regularities of the value function $V(\mathbf{x})$, whose existence has been shown.

3.1. Optimal Switching Regimes.

PROPOSITION 3.2. For any $(\bar{\mathbf{x}}, x_n)$ with $x_n < a(\bar{\mathbf{x}})$,

$$(\alpha - \mathcal{L})f_1(\bar{\mathbf{x}}, a(\bar{\mathbf{x}})) + H(\bar{\mathbf{x}}, a(\bar{\mathbf{x}})) < 0,$$

and for any $(\bar{\mathbf{x}}, x_n)$ with $x_n > a(\bar{\mathbf{x}})$,

$$(\alpha - \mathcal{L})f_1(\bar{\mathbf{x}}, a(\bar{\mathbf{x}})) + H(\bar{\mathbf{x}}, a(\bar{\mathbf{x}})) > 0.$$

Similarly, for any $(\bar{\mathbf{x}}, x_n)$ with $x_n < b(\bar{\mathbf{x}})$,

$$(\alpha - \mathcal{L})f_2(\bar{\mathbf{x}}, b(\bar{\mathbf{x}})) - H(\bar{\mathbf{x}}, b(\bar{\mathbf{x}})) > 0,$$

and for any $(\bar{\mathbf{x}}, x_n)$ with $x_n > b(\bar{\mathbf{x}})$,

$$(\alpha - \mathcal{L})f_2(\bar{\mathbf{x}}, b(\bar{\mathbf{x}})) - H(\bar{\mathbf{x}}, b(\bar{\mathbf{x}})) < 0.$$

From the conditions on H in Assumption 3.1 and the conditions on f_1 and f_2 in Assumption 3.2, this result can be easily seen.

The two boundary curves $A(\bar{\mathbf{x}}), B(\bar{\mathbf{x}})$ are chosen according to the following lemma.

LEMMA 3.3. There exist $A(\bar{\mathbf{x}}) < 0 < B(\bar{\mathbf{x}})$, $\forall \bar{\mathbf{x}} \in \mathbb{R}^{n-1}$ such that the diffusion $\mathbf{M} = (\mathbf{X}_t, P_{\mathbf{x}})$ on \mathbb{D} satisfies

$$(3.11) \quad E_{\xi_1} \left(\int_0^{\tau_0 \wedge \tau_A} e^{-\alpha t} H(\mathbf{X}_t) dt \right) < -2M,$$

$$(3.12) \quad E_{\xi_2} \left(\int_0^{\tau_0 \wedge \tau_B} e^{-\alpha t} H(\mathbf{X}_t) dt \right) > 2M,$$

for some $\xi_1 \in \mathbb{R}^{n-1} \times (A(\bar{\mathbf{x}}), 0)$, and $\xi_2 \in \mathbb{R}^{n-1} \times (0, B(\bar{\mathbf{x}}))$, M being given in Assumption 3.2, and τ_0, τ_A, τ_B denote the hitting times to the graphs of $x_n = 0, A(\bar{\mathbf{x}}), B(\bar{\mathbf{x}})$ respectively.

Proof. Let $R_{\alpha}^{\mathbb{D}}$ be the resolvent operator of an absorbing diffusion \mathbf{D} on the region $\mathbb{D} \subset \mathbb{R}^n$, whose dynamic equation in the interior of \mathbb{D} is given by (1.1). Let $c = c(\bar{\mathbf{x}})$, $\bar{\mathbf{x}} \in \mathbb{R}^{n-1}$ be any bounded continuous function. Since \mathbf{D} is conservative, the hitting probability $E_{\mathbf{x}}(e^{-\alpha \tau_c})$ ($\alpha > 0$) tends to zero as $x_n \rightarrow \pm\infty$. From

$$\begin{aligned} R_{\alpha}^{\mathbb{R}^{n-1} \times (-\infty, c)} 1(\bar{\mathbf{x}}, x_n) &= E_{\mathbf{x}} \left(\int_0^{\tau_c} e^{-\alpha t} dt \right) \\ &= E_{\mathbf{x}} \left(\frac{1}{\alpha} - \frac{1}{\alpha} e^{-\alpha \tau_c} \right), \quad \mathbf{x} = (\bar{\mathbf{x}}, x_n), \end{aligned}$$

we have

$$(3.13) \quad \lim_{x_n \rightarrow -\infty} R_{\alpha}^{\mathbb{R}^{n-1} \times (-\infty, c)} 1(\bar{\mathbf{x}}, x_n) = \frac{1}{\alpha}.$$

By Assumption 3.1, we can choose $l(\bar{\mathbf{x}}) < 0(\bar{\mathbf{x}} \in \mathbb{R}^{n-1})$ uniformly bounded, and

$$H(\mathbf{x}) < -4\alpha M, \quad \forall \mathbf{x} \in \mathbb{R}^{n-1} \times (-\infty, l).$$

Then by (3.13), for each $\bar{\mathbf{x}}_0 \in \mathbb{R}^{n-1}$ there exists $\eta_1(\bar{\mathbf{x}}_0) < l(\bar{\mathbf{x}}_0)$ such that

$$R_\alpha^{\mathbb{R}^{n-1} \times (-\infty, l)} 1(\bar{\mathbf{x}}_0, x_n) > \frac{3}{4\alpha}, \quad x_n \leq \eta_1(\bar{\mathbf{x}}_0).$$

Because the generator of the underlying process is uniformly elliptic, the function $\eta_1(\bar{\mathbf{x}})$ can be chosen uniformly bounded.

Choose a smooth and uniformly bounded function $A(\bar{\mathbf{x}})$ such that $A(\bar{\mathbf{x}}) < \eta_1(\bar{\mathbf{x}})$, $\forall \bar{\mathbf{x}} \in \mathbb{R}^{n-1}$, then again due to the uniform ellipticity of the generator, $A(\bar{\mathbf{x}})$ can be chosen so that

$$R_\alpha^{\mathbb{R}^{n-1} \times (A, l)} 1(\bar{\mathbf{x}}, \eta_1(\bar{\mathbf{x}})) > \frac{1}{2\alpha}, \quad \forall \bar{\mathbf{x}} \in \mathbb{R}^{n-1}.$$

Now the process obtained from \mathbf{M} by killing at $\tau_0 \wedge \tau_A$ coincides with \mathbf{D} on $\mathbb{R}^{n-1} \times (A, 0)$ and we may just pick $\xi_1 = (\bar{\mathbf{x}}, \eta_1(\bar{\mathbf{x}}))$ and have

$$\begin{aligned} E_{\xi_1} \left(\int_0^{\tau_0 \wedge \tau_A} e^{-\alpha t} H(\mathbf{X}_t) dt \right) &= R_\alpha^{\mathbb{R}^{n-1} \times (A, 0)} H(\xi_1) \\ &\leq R_\alpha^{\mathbb{R}^{n-1} \times (A, l)} H(\xi_1) \\ &< -4\alpha M R_\alpha^{\mathbb{R}^{n-1} \times (A, l)} 1(\xi_1) \\ &< -2M. \end{aligned}$$

The second inequality of this Lemma can be proved in a similar way. \square

In the above proof, we could also choose $A(\bar{\mathbf{x}})$ so that

$$(3.14) \quad E_{\mathbf{x}}(e^{-\alpha \tau_{\eta_1}}) \leq \frac{1}{2}, \quad \forall \mathbf{x} = (\bar{\mathbf{x}}, x_n) \in \mathbb{R}^n \text{ and } x_n < A(\bar{\mathbf{x}}).$$

A similar assumption is made on the curve $B(\bar{\mathbf{x}})$.

Consider the solution $V(\mathbf{x})$ of the variational inequality problem (2.3) and (2.4) for the Dirichlet from $(\mathcal{E}, \mathcal{F})$ (3.3) on \mathbb{D} , which is also the value function of the Dynkin game by Theorem 3.1.

PROPOSITION 3.4. *It is not optimal for P_1 to stop the game when $\mathbf{X}_t < 0$ (or equivalently $H(\mathbf{X}_t) < 0$), and it is not optimal for P_2 to stop the game when $\mathbf{X}_t > 0$ (or equivalently $H(\mathbf{X}_t) > 0$).*

Proof. Recall that P_1 tries to minimize his payment. If $H(\mathbf{X}_t) < 0$, the accumulated amount keeps decreasing, and it would be better for P_1 to stop the game when $X_t > 0$ (or equivalently $H(\mathbf{X}_t) > 0$) because the penalty $f_2(\mathbf{X}_t)$ would be smaller in this case. A similar argument can be made for P_2 . \square

PROPOSITION 3.5. *For any starting point $\mathbf{x}_0 = (\bar{\mathbf{x}}_0, x_n) \in \mathbb{R}^n$ of the game, if $x_n \leq A(\bar{\mathbf{x}}_0)$, it is optimal for P_2 to stop the game immediately; and if $x_n \geq B(\bar{\mathbf{x}}_0)$, it is optimal for P_1 to stop the game immediately.*

Proof. We only need to show the first half and the second half can be done in a similar way. If P_2 stops the game immediately when $x_n \leq A(\bar{\mathbf{x}}_0)$, he receives the quantity $-f_1(\mathbf{x}_0)$ from P_1 (indeed he pays money to P_1). Assume the game is not stopped immediately but is stopped at a stopping time σ . If $\sigma \leq \tau_{\eta_1}$, this means

P_2 stops it before the process reaches the curve η_1 which was given in the proof of Lemma 3.3, then the amount P_2 receives is given by

$$\begin{aligned} & E_{\mathbf{x}_0} \left(\int_0^\sigma e^{-\alpha t} H(\mathbf{X}_t) dt \right) - E_{\mathbf{x}_0} (e^{-\alpha \sigma} f_1(\mathbf{X}_\sigma)) \\ & < \frac{-4\alpha M(1 - E_{\mathbf{x}_0}(e^{-\alpha \sigma}))}{\alpha} - E_{\mathbf{x}_0} (e^{-\alpha \sigma} f_1(\mathbf{X}_\sigma)) \\ & = -4M(1 - E_{\mathbf{x}_0}(e^{-\alpha \sigma})) - E_{\mathbf{x}_0} (e^{-\alpha \sigma} f_1(\mathbf{X}_\sigma)) \\ & \leq -f_1(\mathbf{x}_0) \quad \text{by Assumption 3.3.} \end{aligned}$$

Suppose $\sigma > \tau_{\eta_1}$, then the amount P_2 receives is at most

$$E_{\mathbf{x}_0} \left(\int_0^{\tau_{\eta_1}} e^{-\alpha t} H(\mathbf{X}_t) dt \right) + E_{\mathbf{x}_0} (e^{-\alpha \tau_{\eta_1}} V(\mathbf{X}_{\tau_{\eta_1}})),$$

where $V(\cdot)$ is the solution of the variational inequality problem (2.3) and (2.4), hence $|V(\mathbf{x})| \leq M$, $\forall \mathbf{x} \in \mathbb{R}^n$.

Therefore we get

$$\begin{aligned} & E_{\mathbf{x}_0} \left(\int_0^{\tau_{\eta_1}} e^{-\alpha t} H(\mathbf{X}_t) dt \right) + E_{\mathbf{x}_0} (e^{-\alpha \tau_{\eta_1}} V(\mathbf{X}_{\tau_{\eta_1}})) \\ & < -4M(1 - E_{\mathbf{x}_0}(e^{-\alpha \sigma})) + M \\ & \leq -4M \left(1 - \frac{1}{2} \right) + M \quad \text{by the choice of } A \text{ and } \eta_1 \\ & = -M < -f_1(\mathbf{x}_0) \quad \text{since } f_1 \text{ is bounded by } M. \end{aligned}$$

From both cases, we can conclude that it is optimal for P_2 to stop the process immediately if $x_n \leq A(\bar{\mathbf{x}}_0)$. Similarly we can show that it is optimal for P_1 to stop the process immediately if the starting point is $\mathbf{x}_0 = (\bar{\mathbf{x}}_0, x_n) \in \mathbb{R}^n$ with $x_n \geq B(\bar{\mathbf{x}}_0)$. \square

COROLLARY 3.6. *Let $(\hat{\tau}, \hat{\sigma})$ be the saddle point in (3.9), then $\hat{\tau}, \hat{\sigma}$ are finite a.s.*

Proof. Because the two curves $A(\bar{\mathbf{x}}), B(\bar{\mathbf{x}})$ are uniformly bounded, and the generator of the underlying process is uniformly elliptic, if we let τ_A, τ_B be the first hitting time of the last component X_n to the two curves $A(\bar{\mathbf{x}}), B(\bar{\mathbf{x}})$, assuming that the game starts at a point inside $\mathbb{R}^{n-1} \times (A, B)$, then we can conclude that $\tau_A \wedge \tau_B$ is finite a.s., and the result simply follows due to the fact that $\hat{\tau} \wedge \hat{\sigma} \leq \tau_A \wedge \tau_B$ a.s. and beyond the region \mathbb{D} the game is stopped immediately. \square

PROPOSITION 3.7.

1. $-f_1(\bar{\mathbf{x}}, 0) < V(\bar{\mathbf{x}}, 0) < f_2(\bar{\mathbf{x}}, 0)$, $\forall \bar{\mathbf{x}} \in \mathbb{R}^{n-1}$.

Proof. Let $\delta(\bar{\mathbf{x}}) > 0$ be a continuous function such that $-f_1(\bar{\mathbf{x}}, 0) < H(\bar{\mathbf{x}}, -\delta(\bar{\mathbf{x}})) < 0$, $-\delta(\bar{\mathbf{x}}) > A(\bar{\mathbf{x}})$, $\forall \bar{\mathbf{x}} \in \mathbb{R}^{n-1}$, $\sup_{\bar{\mathbf{x}}} |\delta(\bar{\mathbf{x}})| = b_\delta < \infty$, and $b_\delta \rightarrow 0$ as $\delta \rightarrow 0$. Let τ_δ be the hitting time of the process to the curve $\delta(\bar{\mathbf{x}})$, then from (3.9) and for all $\bar{\mathbf{x}} \in \mathbb{R}^{n-1}$, the following holds:

$$\begin{aligned} & V(\bar{\mathbf{x}}, 0) \\ & \geq J_{(\bar{\mathbf{x}}, 0)}(\hat{\tau}, \tau_{-\delta}) \\ & = E_{(\bar{\mathbf{x}}, 0)} \left(\int_0^{\hat{\tau} \wedge \tau_{-\delta}} e^{-\alpha t} H(\mathbf{X}_t) dt \right) \\ & \quad - E_{(\bar{\mathbf{x}}, 0)} (e^{-\alpha \tau_{-\delta}} f_1(\mathbf{X}_{\tau_{-\delta}}) | \tau_{-\delta} < \hat{\tau}) P(\tau_{-\delta} < \hat{\tau}) \end{aligned}$$

$$\begin{aligned}
& +E_{(\bar{\mathbf{x}},0)} \left(e^{-\alpha\hat{\tau}} f_2(\mathbf{X}_{\hat{\tau}}) | \tau_{-\delta} > \hat{\tau} \right) P(\tau_{-\delta} > \hat{\tau}) \\
& > -L_2 b_\delta E_{(\bar{\mathbf{x}},0)} \left(\int_0^{\hat{\tau} \wedge \tau_{-\delta}} e^{-\alpha t} dt \right) \\
& \quad -E_{(\bar{\mathbf{x}},0)} \left(e^{-\alpha\tau_{-\delta}} f_1(\mathbf{X}_{\tau_{-\delta}}) | \tau_{-\delta} < \hat{\tau} \right) P(\tau_{-\delta} < \hat{\tau}) \\
& \quad +\beta E_{(\bar{\mathbf{x}},0)} \left(e^{-\alpha\hat{\tau}} | \tau_{-\delta} > \hat{\tau} \right) P(\tau_{-\delta} > \hat{\tau}) \\
& = -\frac{L_2 b_\delta}{\alpha} \left(1 - E_{(\bar{\mathbf{x}},0)} \left(e^{-\alpha\tau_{-\delta}} | \tau_{-\delta} < \hat{\tau} \right) \right) P(\tau_{-\delta} < \hat{\tau}) \\
& \quad -\frac{L_2 b_\delta}{\alpha} \left(1 - E_{(\bar{\mathbf{x}},0)} \left(e^{-\alpha\hat{\tau}} | \tau_{-\delta} > \hat{\tau} \right) \right) P(\tau_{-\delta} > \hat{\tau}) \\
& \quad -E_{(\bar{\mathbf{x}},0)} \left(e^{-\alpha\tau_{-\delta}} f_1(\mathbf{X}_{\tau_{-\delta}}) | \tau_{-\delta} < \hat{\tau} \right) P(\tau_{-\delta} < \hat{\tau}) \\
& \quad +\beta E_{(\bar{\mathbf{x}},0)} \left(e^{-\alpha\hat{\tau}} | \tau_{-\delta} > \hat{\tau} \right) P(\tau_{-\delta} > \hat{\tau}) \\
& = -f_1(\bar{\mathbf{x}}, 0) \\
& \quad +E_{(\bar{\mathbf{x}},0)} \left(f_1(\bar{\mathbf{x}}, 0) - e^{-\alpha\tau_{-\delta}} f_1(\mathbf{X}_{\tau_{-\delta}}) - \frac{L_2 b_\delta}{\alpha} (1 - e^{-\alpha\tau_{-\delta}}) | \tau_{-\delta} < \hat{\tau} \right) P(\tau_{-\delta} < \hat{\tau}) \\
& \quad +E_{(\bar{\mathbf{x}},0)} \left(f_1(\bar{\mathbf{x}}, 0) + \beta e^{-\alpha\hat{\tau}} - \frac{L_2 b_\delta}{\alpha} (1 - e^{-\alpha\hat{\tau}}) | \tau_{-\delta} > \hat{\tau} \right) P(\tau_{-\delta} > \hat{\tau}) \\
& > -f_1(\bar{\mathbf{x}}, 0) \\
& \quad +E_{(\bar{\mathbf{x}},0)} \left(f_1(\bar{\mathbf{x}}, 0) - e^{-\alpha\tau_{-\delta}} f_1(\mathbf{X}_{\tau_{-\delta}}) - \frac{L_2 b_\delta}{\alpha} (1 - e^{-\alpha\tau_{-\delta}}) | \tau_{-\delta} < \hat{\tau} \right) P(\tau_{-\delta} < \hat{\tau}) \\
& > -f_1(\bar{\mathbf{x}}, 0) \\
& \quad +E_{(\bar{\mathbf{x}},0)} \left(c_1 (1 - e^{-\alpha\tau_{-\delta}}) - \frac{L_2 b_\delta}{\alpha} (1 - e^{-\alpha\tau_{-\delta}}) | \tau_{-\delta} < \hat{\tau} \right) P(\tau_{-\delta} < \hat{\tau}) \\
& = -f_1(\bar{\mathbf{x}}, 0) + E_{(\bar{\mathbf{x}},0)} \left(\left(c_1 - \frac{L_2 b_\delta}{\alpha} \right) (1 - e^{-\alpha\tau_{-\delta}}) | \tau_{-\delta} < \hat{\tau} \right) P(\tau_{-\delta} < \hat{\tau}) \\
& > -f_1(\bar{\mathbf{x}}, 0) \text{ for sufficiently small } b_\delta.
\end{aligned}$$

The second inequality can be proved in a similar way. \square

2. $V(\mathbf{x}) > -f_1(\mathbf{x})$ for $\mathbf{x} \in \mathbb{R}^{n-1} \times (0, B)$ and $V(\mathbf{x}) < f_2(\mathbf{x})$ for $\mathbf{x} \in \mathbb{R}^{n-1} \times (A, 0)$.

Proof. We may just give proof to the first inequality and the second one can be proved analogously. Notice that for $\mathbf{x} \in \mathbb{R}^{n-1} \times (0, B)$,

$$\begin{aligned}
V(\mathbf{x}) & \geq J_{\mathbf{x}}(\hat{\tau}, \tau_0) \\
& = E_{\mathbf{x}} \left(\int_0^{\hat{\tau} \wedge \tau_0} e^{-\alpha t} H(\mathbf{X}_t) dt \right) - E_{\mathbf{x}} \left(e^{-\alpha\tau_0} f_1(\mathbf{X}_{\tau_0}) | \tau_0 < \hat{\tau} \right) P(\tau_0 < \hat{\tau}) \\
& \quad + E_{\mathbf{x}} \left(e^{-\alpha\hat{\tau}} f_2(\mathbf{X}_{\hat{\tau}}) | \tau_0 > \hat{\tau} \right) P(\tau_0 > \hat{\tau}) \\
& > -c_1 E_{\mathbf{x}} \left(e^{-\alpha\tau_0} | \tau_0 < \hat{\tau} \right) P(\tau_0 < \hat{\tau}) \\
& > -c_1 \\
& > -f_1(\mathbf{x}) \text{ by Assumption 3.2.}
\end{aligned}$$

\square

Define the set

$$(3.15) \quad E = \{\mathbf{x} \in \mathbb{R}^n : -f_1(\mathbf{x}) < V(\mathbf{x}) < f_2(\mathbf{x})\},$$

then obviously $(\bar{\mathbf{x}}, 0) \in E$, $\forall \bar{\mathbf{x}} \in \mathbb{R}^{n-1}$. Since P_2 would stop the game once $V(\mathbf{x}) \leq -f_1(\mathbf{x})$ and the instant payoff is $-f_1(\mathbf{x})$, while P_1 would stop the game once $V(\mathbf{x}) \geq f_2(\mathbf{x})$ and the instant payoff is $f_2(\mathbf{x})$, we could write \mathbb{R}^n as a partition:

$$\mathbb{R}^n = E_1 \cup E \cup E_2,$$

where E_1, E_2 were given in Theorem 3.1.

PROPOSITION 3.8. *For each $\mathbf{x} \in E_1$,*

$$(\alpha - \mathcal{L})f_1(\mathbf{x}) + H(\mathbf{x}) \leq 0,$$

and for each $\mathbf{x} \in E_2$,

$$(\alpha - \mathcal{L})f_2(\mathbf{x}) - H(\mathbf{x}) \leq 0.$$

Proof. We only give proof to the first half. We know at the point $\mathbf{x} \in E_1$ it must be true that $V(\mathbf{x}) \leq -f_1(\mathbf{x})$, and it is optimal for P_2 to stop the game immediately. Suppose

$$(\alpha - \mathcal{L})f_1(\mathbf{x}) + H(\mathbf{x}) > 0,$$

then by the smoothness of f_1 , the continuity of H and the Lipschitz condition on $a(\cdot)$ and $b(\cdot)$, we can find a small ball $B_r(\mathbf{x})$ containing the point \mathbf{x} , such that for each $\mathbf{y} \in B_r(\mathbf{x})$,

$$(\alpha - \mathcal{L})f_1(\mathbf{y}) + H(\mathbf{y}) > 0.$$

Consider a policy for P_2 to stop the game at the first exit time of $B_r(\mathbf{x})$, denoted τ_{B_r} . Then by Dynkin's formula, the payoff would be

$$\begin{aligned} J_{\mathbf{x}} &= E_{\mathbf{x}} \int_0^{\tau_{B_r}} e^{-\alpha t} H(\mathbf{X}_t) dt + E_{\mathbf{x}}(e^{-\alpha \tau_{B_r}}(-f_1(\mathbf{X}_{\tau_{B_r}}))) \\ &= E_{\mathbf{x}} \int_0^{\tau_{B_r}} e^{-\alpha t} H(\mathbf{X}_t) dt - f_1(\mathbf{x}) + E_{\mathbf{x}} \int_0^{\tau_{B_r}} e^{-\alpha t} (\alpha - \mathcal{L})f_1(\mathbf{X}_t) dt \\ &= -f_1(\mathbf{x}) + E_{\mathbf{x}} \int_0^{\tau_{B_r}} e^{-\alpha t} [(\alpha - \mathcal{L})f_1(\mathbf{X}_t) + H(\mathbf{X}_t)] dt \\ &> -f_1(\mathbf{x}). \end{aligned}$$

This is a contradiction since P_2 tries to maximize his payoff but we assumed the optimal policy at \mathbf{x} was to stop the game immediately. \square

COROLLARY 3.9. *If $\mathbf{x} = (\bar{\mathbf{x}}, x_n) \in E_1$, then for any point $(\bar{\mathbf{x}}, y)$ with $y < x_n$,*

$$(\alpha - \mathcal{L})f_1(\bar{\mathbf{x}}, y) + H(\bar{\mathbf{x}}, y) < 0.$$

If $\mathbf{x} = (\bar{\mathbf{x}}, x_n) \in E_2$, then for any point $(\bar{\mathbf{x}}, y)$ with $y > x_n$,

$$(\alpha - \mathcal{L})f_2(\bar{\mathbf{x}}, y) - H(\bar{\mathbf{x}}, y) < 0.$$

This can be easily seen from Proposition 3.8 and the properties of f_1, f_2 given in Assumption 3.2.

PROPOSITION 3.10. *If*

$$(\alpha - \mathcal{L})f_1(\bar{\mathbf{x}}, x_n) + H(\bar{\mathbf{x}}, x_n) < 0, \quad \forall \bar{\mathbf{x}} \in \mathbb{R}^{n-1},$$

then it is optimal for P_2 to stop the game immediately. If

$$(\alpha - \mathcal{L})f_2(\bar{\mathbf{x}}, x_n) - H(\bar{\mathbf{x}}, x_n) > 0, \quad \forall \bar{\mathbf{x}} \in \mathbb{R}^{n-1},$$

then it is optimal for P_1 to stop the game immediately.

Proof. Again we give the proof of the first half. Suppose

$$(\alpha - \mathcal{L})f_1(\bar{\mathbf{x}}, x_n) + H(\bar{\mathbf{x}}, x_n) < 0, \quad \forall \bar{\mathbf{x}} \in \mathbb{R}^{n-1},$$

then $x_n < a(\bar{\mathbf{x}})$ where $a(\bar{\mathbf{x}})$ is given in Assumption 3.4. If P_2 stops the game immediately, the payoff is $-f_1(\mathbf{x})$, $\mathbf{x} = (\bar{\mathbf{x}}, x_n)$. Suppose it is optimal for P_2 to stop the game at a stopping time $\sigma > 0$ a.s. We compare σ with the hitting times τ_A and τ_a . Suppose $\sigma \leq \tau_a$, by Dynkin's formula, the payoff for P_2 is

$$\begin{aligned} & E_{\mathbf{x}} \left(\int_0^\sigma e^{-\alpha t} H(\mathbf{X}_t) dt - e^{-\alpha \sigma} f_1(\mathbf{X}_\sigma) | \sigma \leq \tau_a \right) P_{\mathbf{x}}(\sigma \leq \tau_a) \\ &= \left(-f_1(\mathbf{x}) + E_{\mathbf{x}} \left(\int_0^\sigma e^{-\alpha t} (H(\mathbf{X}_t) + (\alpha - \mathcal{L})f_1(\mathbf{X}_t)) dt | \sigma \leq \tau_a \right) \right) P_{\mathbf{x}}(\sigma \leq \tau_a) \\ &< -f_1(\mathbf{x}) P_{\mathbf{x}}(\sigma \leq \tau_a), \end{aligned}$$

because

$$H(\mathbf{y}) + (\alpha - \mathcal{L})f_1(\mathbf{y}) < 0, \quad \forall \mathbf{y} \in \mathbb{R}^{n-1} \times (-\infty, a).$$

Since P_2 tries to maximize his payoff, the result (3.16) tells that along the paths of \mathbf{X}_t , P_2 should not stop the process before these paths touch the curve $(\bar{\mathbf{x}}, a(\bar{\mathbf{x}}))$. However, consider the paths such that $\tau_A < \tau_a$. Because $a(\bar{\mathbf{x}})$ is smooth and uniformly Lipschitz, i.e., it satisfies the interior cone condition, and the underlying process is uniformly elliptic, we have $P(\tau_A < \tau_a) > 0$. But we have shown that the optimal policy for P_2 at the curve $(\bar{\mathbf{x}}, A(\bar{\mathbf{x}}))$ is to stop the game immediately, and this gives a contradiction.

□

By Proposition 3.8 and Proposition 3.10, we can tell

$$\begin{aligned} \mathbb{R}^{n-1} \times (-\infty, a) &\subseteq E_1, \\ \mathbb{R}^{n-1} \times (b, \infty) &\subseteq E_2. \end{aligned}$$

Further, from the proof of Proposition 3.8 we can see that, since

$$(\alpha - \mathcal{L})f_1(\mathbf{x}) + H(\mathbf{x}) > 0, \quad \forall \mathbf{x} \in \mathbb{R}^{n-1} \times (a, b),$$

which implies $V(\mathbf{x}) > -f_1(\mathbf{x})$, it is not optimal for P_2 to stop the game in this region, and also because

$$(\alpha - \mathcal{L})f_2(\mathbf{x}) - H(\mathbf{x}) < 0, \quad \forall \mathbf{x} \in \mathbb{R}^{n-1} \times (a, b),$$

which implies $V(\mathbf{x}) < f_2(\mathbf{x})$, it is not optimal for P_1 to stop the game in this region. Therefore we can tell

$$\mathbb{R}^{n-1} \times (a, b) \subseteq E.$$

By Theorem 3.1, the optimal policy for P_2 is to stop the game at the first hitting time of E_1 , while the optimal policy for P_1 is to stop the game at the first hitting time of E_2 , and because the boundary points of $\mathbb{R}^{n-1} \times (a, b)$ are all regular boundary points, we can conclude that:

COROLLARY 3.11.

$$\begin{aligned}\mathbb{R}^{n-1} \times (-\infty, a] &= E_1, \\ \mathbb{R}^{n-1} \times [b, \infty) &= E_2, \\ \mathbb{R}^{n-1} \times (a, b) &= E.\end{aligned}$$

3.2. Regularities of the Value Function.

PROPOSITION 3.12. V is smooth on $\mathbb{R}^{n-1} \times (a, b)$, and

$$(3.16) \quad \alpha V(\mathbf{x}) - \mathcal{L}V(\mathbf{x}) = H(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^{n-1} \times (a, b),$$

$$(3.17) \quad \alpha V(\mathbf{x}) - \mathcal{L}V(\mathbf{x}) > H(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^{n-1} \times (A, a),$$

$$(3.18) \quad \alpha V(\mathbf{x}) - \mathcal{L}V(\mathbf{x}) < H(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^{n-1} \times (b, B).$$

Proof. We notice that V is H - α harmonic on $\mathbb{R}^{n-1} \times (a, b)$, which by [5] implies the validation of the following equation:

$$(3.19) \quad \mathcal{E}_\alpha(V, u) = (H, u), \quad \forall u \in C_0^{1, \dots, 1}(\mathbb{R}^{n-1} \times (a, b)).$$

The continuity of H implies that V is smooth on the same region, and an integration by parts yields (3.16) on this region. The rest of the proof follows from the fact that $V(\mathbf{x}) = -f_1(\mathbf{x}), \forall \mathbf{x} \in \mathbb{R}^{n-1} \times [A, a]$ and $V(\mathbf{x}) = f_2(\mathbf{x}), \forall \mathbf{x} \in [b, B]$. \square

Now we may extend V to \mathbb{R}^n by letting $V(\mathbf{x}) = -f_1(\mathbf{x}), \forall \mathbf{x} \in \mathbb{R}^{n-1} \times (-\infty, a]$, and $V(\mathbf{x}) = f_2(\mathbf{x}), \forall \mathbf{x} \in \mathbb{R}^{n-1} \times [b, \infty)$.

In what follows we will characterize the properties of V at the boundary curves.

For any point $(\bar{\mathbf{x}}_0, a(\bar{\mathbf{x}}_0))$ on the curve a , let $\delta(\bar{\mathbf{x}}_0)$ be a small positive number, and denote $\delta_1(\bar{\mathbf{x}}), \delta_2(\bar{\mathbf{x}})$ two continuous curves such that $\delta_1(\bar{\mathbf{x}}_0) = \delta_2(\bar{\mathbf{x}}_0) = \delta(\bar{\mathbf{x}}_0) > 0$, and they are chosen in the following way: since \mathbf{X}_t is conservative, we can choose a region $B_{\bar{\mathbf{x}}_0} \subset \mathbb{R}^{n-1}$ containing $\bar{\mathbf{x}}_0$ so that the probability that \mathbf{X}_t leaves $B_{\bar{\mathbf{x}}_0} \times (a - \delta_1, a + \delta_2)$ without hitting $a - \delta_1$ or $a + \delta_2$ is of the order $o(\delta(\bar{\mathbf{x}}_0))$. Because $(\bar{\mathbf{x}}_0, a(\bar{\mathbf{x}}_0))$ is a regular boundary point which satisfies the inner cone condition, we could choose δ_1, δ_2 so that $P_{(\bar{\mathbf{x}}_0, a(\bar{\mathbf{x}}_0))}(\tau_{a-\delta_1} < \tau_{a+\delta_2} \wedge \tau_{a, \mathbb{G}}) = P_{(\bar{\mathbf{x}}_0, a(\bar{\mathbf{x}}_0))}(\tau_{a+\delta_2} < \tau_{a-\delta_1} \wedge \tau_{a, \mathbb{G}}) = \frac{1}{2} - o(\delta(\bar{\mathbf{x}}_0))$, where $\tau_{a-\delta_1}, \tau_{a+\delta_2}$ are the hitting times to the curves $a - \delta_1$ and $a + \delta_2$, respectively, and $\tau_{a, \mathbb{G}}$ is the exit time of the connected region $\mathbb{G} \subset B_{\bar{\mathbf{x}}_0} \times (a - \delta_1, a + \delta_2)$ containing $(\bar{\mathbf{x}}_0, a(\bar{\mathbf{x}}_0))$ but without hitting $a - \delta_1$ or $a + \delta_2$. Notice that $P(\tau_{a, \mathbb{G}} < \tau_{a-\delta_1} \wedge \tau_{a+\delta_2}) = o(\delta(\bar{\mathbf{x}}_0))$. Let $\tau_{a, \delta}$ be the exit time of the region \mathbb{G} , then $\tau_{a, \delta} = \tau_{a, \mathbb{G}} \wedge \tau_{a-\delta_1} \wedge \tau_{a+\delta_2}$ a.s. Furthermore, since V is smooth a.e., on the region $B_{\bar{\mathbf{x}}_0}$ we can make δ_1 and δ_2 satisfy that

$$E_{(\bar{\mathbf{x}}_0, a(\bar{\mathbf{x}}_0))}[e^{-\alpha \tau_{a-\delta_1}} V(\bar{\mathbf{X}}_{\tau_{a, \delta}}) | \tau_{a-\delta_1} < \tau_{a, \mathbb{G}} \wedge \tau_{a+\delta_2}] = V(\bar{\mathbf{x}}_0, (a - \delta)(\bar{\mathbf{x}}_0)) + o(\delta(\bar{\mathbf{x}}_0)),$$

and

$$E_{(\bar{\mathbf{x}}_0, a(\bar{\mathbf{x}}_0))}[e^{-\alpha \tau_{a+\delta_2}} V(\bar{\mathbf{X}}_{\tau_{a, \delta}}) | \tau_{a+\delta_2} < \tau_{a, \mathbb{G}} \wedge \tau_{a-\delta_1}] = V(\bar{\mathbf{x}}_0, (a + \delta)(\bar{\mathbf{x}}_0)) + o(\delta(\bar{\mathbf{x}}_0)).$$

By Assumption 1.1 and the fact that \mathbf{X}_t is uniformly elliptic, we have

$$E_{(\bar{\mathbf{x}}_0, a(\bar{\mathbf{x}}_0))}(\tau_{a, \delta}) = o(\delta(\bar{\mathbf{x}}_0)),$$

(see, e.g., [6], [11]). Further, we notice that

$$1 - E_{(\bar{\mathbf{x}}_0, a(\bar{\mathbf{x}}_0))}(e^{-\alpha \tau_{a, \delta}}) = \alpha E_{(\bar{\mathbf{x}}_0, a(\bar{\mathbf{x}}_0))}\left(\int_0^{\tau_{a, \delta}} e^{-\alpha t} dt\right) \leq \alpha E_{(\bar{\mathbf{x}}_0, a(\bar{\mathbf{x}}_0))}(\tau_{a, \delta}) = o(\delta(\bar{\mathbf{x}}_0)).$$

PROPOSITION 3.13.

$$\frac{\partial V}{\partial x_n}(\bar{\mathbf{x}}, a(\bar{\mathbf{x}})) = -\frac{\partial f_1}{\partial x_n}(\bar{\mathbf{x}}, a(\bar{\mathbf{x}})), \quad \frac{\partial V}{\partial x_n}(\bar{\mathbf{x}}, b(\bar{\mathbf{x}})) = \frac{\partial f_2}{\partial x_n}(\bar{\mathbf{x}}, b(\bar{\mathbf{x}})), \quad \forall \bar{\mathbf{x}} \in \mathbb{R}^{n-1}.$$

Proof. We only prove the first equality, and the second one can be proved analogously. Pick any $\bar{\mathbf{x}}_0 \in \mathbb{R}^{n-1}$ and let $\delta(\bar{\mathbf{x}}_0), \delta_1, \delta_2$ be chosen as discussed a priori, and we borrow the notations $\tau_{a, \mathbb{G}}, \tau_{a-\delta_1}, \tau_{a+\delta_2}, \tau_{a, \delta}$ therein. Since $\hat{\tau} = \tau_{a, \delta} + \hat{\tau} \circ \theta_{\tau_{a, \delta}}$ where θ_t is the shift operator, we get

$$\hat{\tau} \wedge \sigma = \tau_{a, \delta} + (\hat{\tau} \wedge \hat{\sigma}) \circ \theta_{\tau_{a, \delta}},$$

where $\sigma = \tau_{a, \delta} + \hat{\sigma} \circ \theta_{\tau_{a, \delta}}$. Therefore by (3.9)

$$\begin{aligned} V(\bar{\mathbf{x}}_0, a(\bar{\mathbf{x}}_0)) &\geq J_{(\bar{\mathbf{x}}_0, a(\bar{\mathbf{x}}_0))}(\hat{\tau}, \sigma) \\ &= E_{(\bar{\mathbf{x}}_0, a(\bar{\mathbf{x}}_0))}\left(\int_0^{\tau_{a, \delta}} e^{-\alpha t} H(\mathbf{X}_t) dt\right) \\ &\quad + E_{(\bar{\mathbf{x}}_0, a(\bar{\mathbf{x}}_0))}(e^{-\alpha \tau_{a-\delta_1}} V(\mathbf{X}_{\tau_{a, \delta}}) | \tau_{a-\delta_1} < \tau_{a+\delta_2} \wedge \tau_{a, \mathbb{G}}) P_{(\bar{\mathbf{x}}_0, a(\bar{\mathbf{x}}_0))}(\tau_{a-\delta_1} < \tau_{a+\delta_2} \wedge \tau_{a, \mathbb{G}}) \\ &\quad + E_{(\bar{\mathbf{x}}_0, a(\bar{\mathbf{x}}_0))}(e^{-\alpha \tau_{a+\delta_2}} V(\mathbf{X}_{\tau_{a, \delta}}) | \tau_{a+\delta_2} < \tau_{a-\delta_1} \wedge \tau_{a, \mathbb{G}}) P_{(\bar{\mathbf{x}}_0, a(\bar{\mathbf{x}}_0))}(\tau_{a+\delta_2} < \tau_{a-\delta_1} \wedge \tau_{a, \mathbb{G}}) \\ &\quad + E_{(\bar{\mathbf{x}}_0, a(\bar{\mathbf{x}}_0))}(e^{-\alpha \tau_{a, \mathbb{G}}} V(\mathbf{X}_{\tau_{a, \delta}}) | \tau_{a, \mathbb{G}} < \tau_{a-\delta_1} \wedge \tau_{a+\delta_2}) P_{(\bar{\mathbf{x}}_0, a(\bar{\mathbf{x}}_0))}(\tau_{a, \mathbb{G}} < \tau_{a-\delta_1} \wedge \tau_{a+\delta_2}), \end{aligned} \quad (3.20)$$

due to the fact that $V(\bar{\mathbf{x}}, a(\bar{\mathbf{x}}) \pm \delta(\bar{\mathbf{x}})) = J_{(\bar{\mathbf{x}}, a(\bar{\mathbf{x}}) \pm \delta(\bar{\mathbf{x}}))}(\hat{\tau}, \hat{\sigma})$. Since V is bounded, (3.20) further yields

$$\begin{aligned} &(V(\bar{\mathbf{x}}_0, a(\bar{\mathbf{x}}_0) + \delta(\bar{\mathbf{x}}_0)) - V(\bar{\mathbf{x}}_0, a(\bar{\mathbf{x}}_0))) P_{(\bar{\mathbf{x}}_0, a(\bar{\mathbf{x}}_0))}(\tau_{a+\delta_2} < \tau_{a-\delta_1} \wedge \tau_{a, \mathbb{G}}) \\ &\leq (V(\bar{\mathbf{x}}_0, a(\bar{\mathbf{x}}_0)) - V(\bar{\mathbf{x}}_0, a(\bar{\mathbf{x}}_0) - \delta(\bar{\mathbf{x}}_0))) P_{(\bar{\mathbf{x}}_0, a(\bar{\mathbf{x}}_0))}(\tau_{a-\delta_1} < \tau_{a+\delta_2} \wedge \tau_{a, \mathbb{G}}) \\ &\quad - E_{(\bar{\mathbf{x}}_0, a(\bar{\mathbf{x}}_0))}\left(\int_0^{\tau_{a, \delta}} e^{-\alpha t} H(\mathbf{X}_t) dt\right) + o(\delta(\bar{\mathbf{x}}_0)). \end{aligned}$$

By Assumption 3.1, it is obvious that H is bounded on $\mathbb{R}^{n-1} \times [A, 0]$, i.e., there exists $U > 0$ such that $|H(\mathbf{x})| \leq U$, $\forall \mathbf{x} \in \mathbb{R}^{n-1} \times [A, 0]$. Then we have

$$\begin{aligned} (3.21) \quad &(V(\bar{\mathbf{x}}_0, a(\bar{\mathbf{x}}_0) + \delta(\bar{\mathbf{x}}_0)) - V(\bar{\mathbf{x}}_0, a(\bar{\mathbf{x}}_0))) P_{(\bar{\mathbf{x}}_0, a(\bar{\mathbf{x}}_0))}(\tau_{a+\delta_2} < \tau_{a-\delta_1} \wedge \tau_{a, \mathbb{G}}) \\ &\leq (V(\bar{\mathbf{x}}_0, a(\bar{\mathbf{x}}_0)) - V(\bar{\mathbf{x}}_0, a(\bar{\mathbf{x}}_0) - \delta(\bar{\mathbf{x}}_0))) P_{(\bar{\mathbf{x}}_0, a(\bar{\mathbf{x}}_0))}(\tau_{a-\delta_1} < \tau_{a+\delta_2} \wedge \tau_{a, \mathbb{G}}) \\ &\quad + U E_{(\bar{\mathbf{x}}_0, a(\bar{\mathbf{x}}_0))}\left(\int_0^{\tau_{a, \delta}} e^{-\alpha t} dt\right) + o(\delta(\bar{\mathbf{x}}_0)) \\ &= (V(\bar{\mathbf{x}}_0, a(\bar{\mathbf{x}}_0)) - V(\bar{\mathbf{x}}_0, a(\bar{\mathbf{x}}_0) - \delta(\bar{\mathbf{x}}_0))) P_{(\bar{\mathbf{x}}_0, a(\bar{\mathbf{x}}_0))}(\tau_{a-\delta_1} < \tau_{a+\delta_2} \wedge \tau_{a, \mathbb{G}}) \\ &\quad + \frac{U}{\alpha} (1 - E_{(\bar{\mathbf{x}}_0, a(\bar{\mathbf{x}}_0))}(e^{-\alpha \tau_{a, \delta}})) + o(\delta(\bar{\mathbf{x}}_0)). \end{aligned}$$

Divide both sides of (3.21) by $\delta(\bar{\mathbf{x}}_0)$ and let $\delta(\bar{\mathbf{x}}_0) \rightarrow 0$, we get

$$\begin{aligned} & \lim_{\delta(\bar{\mathbf{x}}_0) \rightarrow 0+} \frac{V(\bar{\mathbf{x}}_0, a(\bar{\mathbf{x}}_0) + \delta(\bar{\mathbf{x}}_0)) - V(\bar{\mathbf{x}}_0, a(\bar{\mathbf{x}}_0))}{\delta(\bar{\mathbf{x}}_0)} \\ & \leq \lim_{\delta(\bar{\mathbf{x}}_0) \rightarrow 0+} \frac{V(\bar{\mathbf{x}}_0, a(\bar{\mathbf{x}}_0)) - V(\bar{\mathbf{x}}_0, a(\bar{\mathbf{x}}_0) - \delta(\bar{\mathbf{x}}_0))}{\delta(\bar{\mathbf{x}}_0)} \\ & = - \lim_{\delta(\bar{\mathbf{x}}_0) \rightarrow 0+} \frac{f_1(\bar{\mathbf{x}}_0, a(\bar{\mathbf{x}}_0)) - f_1(\bar{\mathbf{x}}_0, a(\bar{\mathbf{x}}_0) - \delta(\bar{\mathbf{x}}_0))}{\delta(\bar{\mathbf{x}}_0)}. \end{aligned}$$

On the other hand, since

$$-f_1(\bar{\mathbf{x}}_0, a(\bar{\mathbf{x}}_0) + \delta(\bar{\mathbf{x}}_0)) - (-f_1(\bar{\mathbf{x}}_0, a(\bar{\mathbf{x}}_0))) < V(\bar{\mathbf{x}}_0, a(\bar{\mathbf{x}}_0) + \delta(\bar{\mathbf{x}}_0)) - V(\bar{\mathbf{x}}_0, a(\bar{\mathbf{x}}_0)),$$

which means

$$\begin{aligned} & - \lim_{\delta(\bar{\mathbf{x}}_0) \rightarrow 0+} \frac{f_1(\bar{\mathbf{x}}_0, a(\bar{\mathbf{x}}_0) + \delta(\bar{\mathbf{x}}_0)) - f_1(\bar{\mathbf{x}}_0, a(\bar{\mathbf{x}}_0))}{\delta(\bar{\mathbf{x}}_0)} \\ & \leq \lim_{\delta(\bar{\mathbf{x}}_0) \rightarrow 0+} \frac{V(\bar{\mathbf{x}}_0, a(\bar{\mathbf{x}}_0) + \delta(\bar{\mathbf{x}}_0)) - V(\bar{\mathbf{x}}_0, a(\bar{\mathbf{x}}_0))}{\delta(\bar{\mathbf{x}}_0)}, \end{aligned}$$

it can be concluded that

$$\frac{\partial V}{\partial x_n}(\bar{\mathbf{x}}_0, a(\bar{\mathbf{x}}_0)) = -\frac{\partial f_1}{\partial x_n}(\bar{\mathbf{x}}_0, a(\bar{\mathbf{x}}_0))$$

by the differentiability of $-f_1$. \square

Let $\partial V_{\mathbf{u}}$ denote the directional derivative along a vector $\mathbf{u} \in \mathbb{R}^n$. If $\mathbf{u} \neq (\bar{\mathbf{u}}, \partial a_{\bar{\mathbf{u}}}(\bar{\mathbf{x}}))$ (or $\mathbf{u} \neq (\bar{\mathbf{u}}, \partial b_{\bar{\mathbf{u}}}(\bar{\mathbf{x}}))$), by repeating the above proof we can show that $\partial V_{\mathbf{u}}(\bar{\mathbf{x}}, a(\bar{\mathbf{x}})) = -\partial f_{1\mathbf{u}}(\bar{\mathbf{x}}, a(\bar{\mathbf{x}}))$, $\partial V_{\mathbf{u}}(\bar{\mathbf{x}}, b(\bar{\mathbf{x}})) = \partial f_{2\mathbf{u}}(\bar{\mathbf{x}}, b(\bar{\mathbf{x}}))$, $\forall \bar{\mathbf{x}} \in \mathbb{R}^{n-1}$. If \mathbf{u} is in the direction of $(\bar{\mathbf{u}}, \partial a_{\bar{\mathbf{u}}}(\bar{\mathbf{x}}))$ (or $(\bar{\mathbf{u}}, \partial b_{\bar{\mathbf{u}}}(\bar{\mathbf{x}}))$), we have

$$\begin{aligned} \partial V_{\mathbf{u}}(\bar{\mathbf{x}}, a(\bar{\mathbf{x}})) &= \lim_{\lambda \rightarrow 0} \frac{V(\bar{\mathbf{x}} + \lambda \bar{\mathbf{u}}, a(\bar{\mathbf{x}} + \lambda \bar{\mathbf{u}})) - V(\bar{\mathbf{x}}, a(\bar{\mathbf{x}}))}{\lambda} \\ &= \lim_{\lambda \rightarrow 0} \frac{-f_1(\bar{\mathbf{x}} + \lambda \bar{\mathbf{u}}, a(\bar{\mathbf{x}} + \lambda \bar{\mathbf{u}})) - (-f_1(\bar{\mathbf{x}}, a(\bar{\mathbf{x}})))}{\lambda} \\ &= -\partial f_{1\mathbf{u}}(\bar{\mathbf{x}}, a(\bar{\mathbf{x}})), \end{aligned}$$

and by a similar proof, $\partial V_{\mathbf{u}}(\bar{\mathbf{x}}, b(\bar{\mathbf{x}})) = \partial f_{2\mathbf{u}}(\bar{\mathbf{x}}, b(\bar{\mathbf{x}}))$, $\forall \bar{\mathbf{x}} \in \mathbb{R}^{n-1}$.

Since V satisfies (3.16), where H is continuous and \mathcal{L} is uniformly elliptic, and $V = -f_1$ on $\mathbb{R}^{n-1} \times (-\infty, a]$ ($V = f_2$ on $\mathbb{R}^{n-1} \times [b, \infty)$), it can be seen that $\lim_{\lambda \rightarrow 0} \partial_{\mathbf{u}} V((\bar{\mathbf{x}}, a(\bar{\mathbf{x}})) + \lambda \mathbf{u})$ exists for any directional vector \mathbf{u} (similar result is obtained on the curve of b). Therefore, $\partial_{\mathbf{u}} V$ is continuous everywhere.

As a summary we have the following theorem:

THEOREM 3.14. *There exist unique smooth and uniformly Lipschitz functions $a(\bar{\mathbf{x}}), b(\bar{\mathbf{x}})$ such that $A(\bar{\mathbf{x}}) < a(\bar{\mathbf{x}}) < 0 < b(\bar{\mathbf{x}}) < B(\bar{\mathbf{x}})$ and*

$$-f_1(\mathbf{x}) < V(\mathbf{x}) < f_2(\mathbf{x}), \quad \forall \mathbf{x} \in \mathbb{R}^{n-1} \times (a, b),$$

$$V(\mathbf{x}) = -f_1(\mathbf{x}), \quad \forall \mathbf{x} \in \mathbb{R}^{n-1} \times (-\infty, a],$$

$$V(\mathbf{x}) = f_2(\mathbf{x}), \quad \forall \mathbf{x} \in \mathbb{R}^{n-1} \times [b, \infty),$$

$$\partial V_{\mathbf{u}}(\bar{\mathbf{x}}, a(\bar{\mathbf{x}})) = -\partial f_{1\mathbf{u}}(\bar{\mathbf{x}}, a(\bar{\mathbf{x}})), \quad \partial V_{\mathbf{u}}(\bar{\mathbf{x}}, b(\bar{\mathbf{x}})) = \partial f_{2\mathbf{u}}(\bar{\mathbf{x}}, b(\bar{\mathbf{x}})), \quad \forall \bar{\mathbf{x}} \in \mathbb{R}^{n-1},$$

where \mathbf{u} is any directional vector.

Furthermore V is $C^{1,\dots,1,1}$ on \mathbb{R}^n , $C^{2,\dots,2}$ on $\mathbb{R}^{n-1} \times (a, b) \cup \mathbb{R}^{n-1} \times (-\infty, a) \cup \mathbb{R}^{n-1} \times (b, \infty)$ and

$$\begin{aligned} \alpha V(\mathbf{x}) - \mathcal{L}V(\mathbf{x}) &= H(\mathbf{x}), \quad \forall \mathbf{x} \in \mathbb{R}^{n-1} \times (a, b), \\ \alpha V(\mathbf{x}) - \mathcal{L}V(\mathbf{x}) &> H(\mathbf{x}), \quad \forall \mathbf{x} \in \mathbb{R}^{n-1} \times (-\infty, a), \\ \alpha V(\mathbf{x}) - \mathcal{L}V(\mathbf{x}) &< H(\mathbf{x}), \quad \forall \mathbf{x} \in \mathbb{R}^{n-1} \times (b, \infty), \end{aligned}$$

where \mathcal{L} is given in (3.2).

4. The Multi-Dimensional Stochastic Singular Control Problem. Define $h(\mathbf{x}), W(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^n$, as follows:

$$(4.1) \quad h(\bar{\mathbf{x}}, y) = \int_0^y H(\bar{\mathbf{x}}, u) du + C(\bar{\mathbf{x}}),$$

$$(4.2) \quad W(\bar{\mathbf{x}}, y) = \int_{a(\bar{\mathbf{x}})}^y V(\bar{\mathbf{x}}, u) du, \quad \bar{\mathbf{x}} \in \mathbb{R}^{n-1}, \quad y \in \mathbb{R},$$

where $C(\bar{\mathbf{x}})$ is a function of $\bar{\mathbf{x}}$ such that

$$\lim_{y \rightarrow a(\bar{\mathbf{x}})+} \alpha W(\bar{\mathbf{x}}, y) - \mathcal{L}W(\bar{\mathbf{x}}, y) - h(\bar{\mathbf{x}}, y) = 0,$$

then $h(\bar{\mathbf{x}}, y)$ and $W(\bar{\mathbf{x}}, y)$ satisfy the following:

THEOREM 4.1. W is $C^{2,\dots,2}$ on \mathbb{R}^n and there exist unique, smooth and uniformly Lipschitz functions $a(\bar{\mathbf{x}}), b(\bar{\mathbf{x}})$ such that $A(\bar{\mathbf{x}}) < a(\bar{\mathbf{x}}) < 0 < b(\bar{\mathbf{x}}) < B(\bar{\mathbf{x}})$ and

$$\begin{aligned} \alpha W(\mathbf{x}) - \mathcal{L}W(\mathbf{x}) &= h(\mathbf{x}), \quad \forall \mathbf{x} \in \mathbb{R}^{n-1} \times (a, b), \\ \alpha W(\mathbf{x}) - \mathcal{L}W(\mathbf{x}) &< h(\mathbf{x}), \quad \forall \mathbf{x} \in \mathbb{R}^{n-1} \times (-\infty, a) \cup \mathbb{R}^{n-1} \times (b, \infty), \\ -f_1(\mathbf{x}) &< \frac{\partial}{\partial x_n} W(\mathbf{x}) < f_2(\mathbf{x}), \quad \forall \mathbf{x} \in \mathbb{R}^{n-1} \times (a, b), \\ \frac{\partial}{\partial x_n} W(\mathbf{x}) &= -f_1(\mathbf{x}), \quad \forall \mathbf{x} \in \mathbb{R}^{n-1} \times (-\infty, a], \\ \frac{\partial}{\partial x_n} W(\mathbf{x}) &= f_2(\mathbf{x}), \quad \forall \mathbf{x} \in \mathbb{R}^{n-1} \times [b, \infty), \end{aligned}$$

and

$$\begin{aligned} \frac{\partial^2}{\partial x_n \partial x_k} W(\bar{\mathbf{x}}, a(\bar{\mathbf{x}})) &= -\frac{\partial f_1}{\partial x_k}(\bar{\mathbf{x}}, a(\bar{\mathbf{x}})), \\ \frac{\partial^2}{\partial x_n \partial x_k} W(\bar{\mathbf{x}}, b(\bar{\mathbf{x}})) &= \frac{\partial f_2}{\partial x_k}(\bar{\mathbf{x}}, b(\bar{\mathbf{x}})), \quad \forall \bar{\mathbf{x}} \in \mathbb{R}^{n-1}, \quad 1 \leq k \leq n. \end{aligned}$$

We first need a lemma.

LEMMA 4.2. *The function $\alpha W(\mathbf{x}) - \mathcal{L}W(\mathbf{x})$ is continuous.*

Proof. This result obviously holds for $\mathbf{x} \in \mathbb{R}^{n-1} \times (-\infty, a) \cup \mathbb{R}^{n-1} \times (a, b) \cup \mathbb{R}^{n-1} \times (b, \infty)$. On the curves $a(\bar{\mathbf{x}})$ and $b(\bar{\mathbf{x}})$, W is twice continuously differentiable along the x_n direction by (4.2). The only term that seems not to be continuous in this function is $\mathcal{L}W(\mathbf{x})$, which involves the first and second derivative with respect to each variable.

Denote $\partial_{x_k} W$ the directional derivative along $x_k, 1 \leq k \leq n-1$, then by (4.2) we have the following:

$$\partial_{x_k} W(\bar{\mathbf{x}}, y) = \int_{a(\bar{\mathbf{x}})}^y \partial_{x_k} V(\bar{\mathbf{x}}, u) du - V(\bar{\mathbf{x}}, a(\bar{\mathbf{x}})) \partial_{x_k} a(\bar{\mathbf{x}}).$$

Notice that V is $C^{1,1,\dots,1}$ on \mathbb{R}^n , so $\partial_{x_k} W(\bar{\mathbf{x}}, y) = \partial_{x_k} W(\bar{\mathbf{x}})$ is continuous in $\bar{\mathbf{x}}$. Now consider

$$\begin{aligned} \frac{\partial^2 W}{\partial x_k^2}(\bar{\mathbf{x}}, y) &= \int_{a(\bar{\mathbf{x}})}^y \frac{\partial^2 V}{\partial x_k^2}(\bar{\mathbf{x}}, u) du - 2 \partial_{x_k} V(\bar{\mathbf{x}}, a(\bar{\mathbf{x}})) \cdot \partial_{x_k} a(\bar{\mathbf{x}}) \\ &\quad - \partial_{x_n} V(\bar{\mathbf{x}}, a(\bar{\mathbf{x}})) \cdot (\partial_{x_k} a(\bar{\mathbf{x}}))^2 - V(\bar{\mathbf{x}}, a(\bar{\mathbf{x}})) \frac{\partial^2 a}{\partial x_k^2}(\bar{\mathbf{x}}). \end{aligned}$$

Because the two curves $a(\bar{\mathbf{x}})$ and $b(\bar{\mathbf{x}})$ have zero Lebesgue measure, and the functions $\partial_{x_k} V$, $\partial_{x_k} a$ and $\frac{\partial^2 a}{\partial x_k^2}$ are all continuous, we conclude that $\frac{\partial^2 W}{\partial x_k^2}$ is continuous, $1 \leq k \leq n-1$. The continuity of $\frac{\partial^2 W}{\partial x_i \partial x_j}, i \neq j$, can be proved in a similar manner. Combined with previous argument that $\frac{\partial^2 W}{\partial x_n^2}$ is continuous, this lemma is proved. \square

REMARK 4.1. *Since $(\alpha - \mathcal{L})W$ is continuous, and the functions V and H are continuous too, we know that the function $h(\bar{\mathbf{x}})$ in (4.1) is continuous, hence the continuity of $C(\bar{\mathbf{x}})$ in (4.1).*

Proof. [Proof of Theorem 4.1] For fixed $\bar{\mathbf{x}}$, consider the function

$$U(y) = \alpha W(\bar{\mathbf{x}}, y) - \mathcal{L}W(\bar{\mathbf{x}}, y) - h(\bar{\mathbf{x}}, y)$$

with

$$U'(y) = \alpha V(\bar{\mathbf{x}}, y) - \mathcal{L}V(\bar{\mathbf{x}}, y) - H(\bar{\mathbf{x}}, y),$$

and we know $U(a(\bar{\mathbf{x}})) = 0$. Notice that $U'(y) = 0$ for $a(\bar{\mathbf{x}}) < y < b(\bar{\mathbf{x}})$; $U'(y) > 0$ for $y < a(\bar{\mathbf{x}})$; $U'(y) < 0$ for $y > b(\bar{\mathbf{x}})$, and by Lemma 4.2 the function $U(y)$ is continuous, it can be seen that

$$\alpha W(\bar{\mathbf{x}}, y) - \mathcal{L}W(\bar{\mathbf{x}}, y) < h(\bar{\mathbf{x}}, y), \text{ for } y < a(\bar{\mathbf{x}}) \text{ or } y > b(\bar{\mathbf{x}}).$$

The rest of the proof is obvious. \square

The result of Theorem 4.1 gives conditions to the solution of the stochastic singular control problem (1.3) and (1.5) (see, e.g., [15]), where the holding cost $h(\cdot)$ is given in (4.1) and the boundary penalty costs $f_1(\cdot), f_2(\cdot)$ are given in Assumption 3.1.

We call a quadruplet $\mathcal{S} = (S, \mathbf{X}_t, A_t^{(1)}, A_t^{(2)})$ ($\mathcal{S} = (A_t^{(1)}, A_t^{(2)})$ for simplicity) admissible policy if the following conditions are satisfied:

ASSUMPTION 4.1.

[1] S is a compact region given in the form $\mathbb{R}^{n-1} \times [\beta, \gamma]$ where $\beta(\bar{\mathbf{x}}), \gamma(\bar{\mathbf{x}})$ are continuous functions of $\bar{\mathbf{x}} \in \mathbb{R}^{n-1}$ with $\beta(\bar{\mathbf{x}}) < \gamma(\bar{\mathbf{x}})$.

[2] There is a filtered measurable space $(\Omega, \{\mathcal{F}_t\}_{t \geq 0})$ subject to usual conditions and a probability measure $\{P_{\mathbf{x}}\}_{\mathbf{x} \in S}$ on it such that

$\{\mathbf{X}_t\}_{t \geq 0}$ is an $\{\mathcal{F}_t\}$ -adapted process, and

$\{A_t^{(1)}, A_t^{(2)}\}_{t \geq 0}$ are $\{\mathcal{F}_t\}$ -adapted right continuous processes with bounded variation such that

$$(4.3) E_{\mathbf{x}} \left(\int_{0-}^{\infty} e^{-\alpha t} dA_t^{(1)} \right) < \infty, E_{\mathbf{x}} \left(\int_{0-}^{\infty} e^{-\alpha t} dA_t^{(2)} \right) < \infty, \forall \mathbf{x} \in S,$$

and $A_t^{(1)} - A_t^{(2)}$ is the minimal decomposition of a bounded variation process into a difference of two increasing processes.

[3] There are $\{\mathcal{F}_t\}$ -adapted independent Brownian motions B_{1t}, \dots, B_{mt} ($m \geq n$) starting at the origin under $P_{\mathbf{x}}$ for any $\mathbf{x} \in S$ such that the following controlled diffusion $\mathbf{X}_t = (X_{1t}, \dots, X_{nt})$

$$(4.4) \quad \begin{aligned} dX_{1t} &= \mu_1 dt + \sigma_{11} dB_{1t} + \dots + \sigma_{1m} dB_{mt}, \\ &\vdots \\ dX_{nt} &= \mu_n dt + \sigma_{n1} dB_{1t} + \dots + \sigma_{nm} dB_{mt} + dA_t^{(1)} - dA_t^{(2)}, \\ \mathbf{X}_0 &= \mathbf{x}, \end{aligned}$$

holds $P_{\mathbf{x}}$ -a.s., $\forall \mathbf{x} \in S$. Furthermore we assume

$$P_{\mathbf{x}}(\mathbf{X}_t \in S, \forall t \geq 0) = 1, \quad \forall \mathbf{x} \in S.$$

REMARK 4.2. The probability space Ω with the filtration $\{\mathcal{F}_t\}$ is not fixed a priori. It is part of an admissible policy. The filtration $\{\mathcal{F}_t\}$ is assumed to be right continuous and \mathcal{F}_0 is assumed to contain every $P_{\mathbf{x}}$ -negligible set for any $\mathbf{x} \in S$.

PROPOSITION 4.3. Both $A_t^{(1)}$ and $A_t^{(2)}$ are nontrivial in the sense that for any $T > 0$,

$$P_{\mathbf{x}}(A_t^{(i)} = A_0^{(i)}, \quad \forall t \in [0, T]) = 0, \quad \forall \mathbf{x} \in S, \quad i = 1, 2.$$

Proof. If both $A_t^{(1)}$ and $A_t^{(2)}$ are trivial, \mathbf{X}_t will hit every open region of positive Lebesgue measure in \mathbb{R}^n with positive probability, but this is a contradiction since \mathbf{X}_t is concentrated on S . If either $A_t^{(1)}$ or $A_t^{(2)}$ is trivial, \mathbf{X}_t can not be concentrated on S which again is a contradiction. \square

Define the following notations:

$$\begin{aligned} \Delta A_t^{(i)} &= A_t^{(i)} - A_{t-}^{(i)}, \quad t \geq 0, i = 1, 2, \\ \Delta \mathbf{X}_t &= \mathbf{X}_t - \mathbf{X}_{t-}, \quad t \geq 0, \\ \Delta W(\mathbf{X}_t) &= W(\mathbf{X}_t) - W(\mathbf{X}_{t-}), \quad t \geq 0. \end{aligned}$$

Then due to the fact that $A_t^{(1)}, A_t^{(2)}$ are the minimal decomposition of a bounded variation process into a difference of two increasing processes, $\Delta A_t^{(1)} \cdot \Delta A_t^{(2)} = 0, \forall t \geq 0$. By convention we let

$$B_{1t} = \dots = B_{nt} = 0, \quad A_t^{(1)} = A_t^{(2)} = 0, \quad \forall t < 0,$$

so that

$$\Delta A_0^{(i)} = A_0^{(i)}, \quad i = 1, 2, \quad \mathbf{X}_0 = \mathbf{x} \quad P_{\mathbf{x}} \text{ a.s.}, \quad \mathbf{x} \in S.$$

Notice that the integrals in (4.3) involve the possible jumps at time 0 so that they are the sum of the integrals over $(0, \infty)$ as well as $A_0^{(i)}, i = 1, 2$. What is more, the jump only happens to the x_n coordinate. In what follows, we use $A_t^{(i),c} (i = 1, 2)$ to denote the continuous part of the processes $A_t^{(i)}, i = 1, 2$.

THEOREM 4.4. Let $h(\mathbf{x}), W(\mathbf{x})$ satisfy the conditions of Theorem 4.1 and $k_S(\mathbf{x})$ be given by the following

$$(4.5) \quad k_S(\mathbf{x}) = E_{\mathbf{x}} \left(\int_0^\infty e^{-\alpha t} h(\mathbf{X}_t) dt \right)$$

$$\begin{aligned}
& + E_{\mathbf{x}} \left(\int_0^\infty e^{-\alpha t} \left(f_1(\mathbf{X}_t) dA_t^{(1),c} + f_2(\mathbf{X}_t) dA_t^{(2),c} \right) \right) \\
& + E_{\mathbf{x}} \left(\sum_{0 \leq t < \infty} e^{-\alpha t} \left(\int_{X_{nt-}}^{X_{nt-} + \Delta A_t^{(1)}} f_1(\mathbf{X}_t) dy \right. \right. \\
& \quad \left. \left. + \int_{X_{nt-} - \Delta A_t^{(2)}}^{X_{nt-}} f_2(\mathbf{X}_t) dy \right) \right),
\end{aligned}$$

where f_1, f_2 satisfy Assumptions 3.2, 3.3 and 3.4, then

1. For any admissible policy \mathcal{S} , $W(\mathbf{x}) \leq k_{\mathcal{S}}(\mathbf{x})$, $\forall \mathbf{x} \in \mathbb{R}^n$.
2. $W(\mathbf{x}) = k_{\mathcal{S}}(\mathbf{x})$, $\forall \mathbf{x} \in \mathbb{R}^n$, if and only if $S = \mathbb{R}^{n-1} \times [a, b]$, where $a(\bar{\mathbf{x}}), b(\bar{\mathbf{x}})$ are given in Theorem 4.1, the process \mathbf{X}_t is the reflecting diffusion on S , and $\mathcal{S} = (A_t^{(1)}, A_t^{(2)})$ where $A_t^{(1)}$ increases only when \mathbf{X}_t is on the boundary $(\bar{\mathbf{x}}, a(\bar{\mathbf{x}}))$ and $A_t^{(2)}$ increases only when \mathbf{X}_t is on the boundary $(\bar{\mathbf{x}}, b(\bar{\mathbf{x}}))$, $\forall \bar{\mathbf{x}} \in \mathbb{R}^{n-1}$.

REMARK 4.3. The cost function consists of several parts. The first integral in (4.5) is the holding cost. The second integral is a control cost associated with the increment of controls $A_t^{(i)}$ ($i = 1, 2$) in the continuous part. The last integral is a control cost associated with the jumps in $A_t^{(i)}$, $i = 1, 2$ (or equivalently jumps in \mathbf{X}_t). We further extend $k_{\mathcal{S}}(\mathbf{x})$ outside the region $\mathbb{R}^{n-1} \times [\beta, \gamma]$ for two continuous functions $\beta(\bar{\mathbf{x}}) < \gamma(\bar{\mathbf{x}})$, $\forall \bar{\mathbf{x}} \in \mathbb{R}^{n-1}$ as the following:

$$(4.6) k_{\mathcal{S}}(\mathbf{x}) = k_{\mathcal{S}}(\bar{\mathbf{x}}, \beta(\bar{\mathbf{x}})) + \int_{x_n}^{\beta(\bar{\mathbf{x}})} f_1(\bar{\mathbf{x}}, u) du, \quad \forall \mathbf{x} = (\bar{\mathbf{x}}, x_n) \in \mathbb{R}^{n-1} \times (-\infty, \beta),$$

$$(4.7) k_{\mathcal{S}}(\mathbf{x}) = k_{\mathcal{S}}(\bar{\mathbf{x}}, \gamma(\bar{\mathbf{x}})) + \int_{\gamma(\bar{\mathbf{x}})}^{x_n} f_2(\bar{\mathbf{x}}, u) du, \quad \forall \mathbf{x} = (\bar{\mathbf{x}}, x_n) \in \mathbb{R}^{n-1} \times (\gamma, \infty),$$

and we are looking for an admissible control \mathcal{S} such that

$$(4.8) \quad W^*(\mathbf{x}) = \inf_{\mathcal{S} \in \mathbb{S}} k_{\mathcal{S}}(\mathbf{x}), \quad \forall \mathbf{x} \in \mathbb{R}^n,$$

where \mathbb{S} is the set of all admissible control policies.

Proof. [Proof of Theorem 4.4]

1. Consider the diffusion given in (4.4) with $\mathbf{x} \in S$. Applying the generalized Ito formula to $e^{-\alpha t} W(\mathbf{X}_t)$ (see [10]) yields

$$\begin{aligned}
(4.9) \quad e^{-\alpha t} W(\mathbf{X}_t) &= W(\mathbf{x}) - \alpha \int_0^t e^{-\alpha s} W(\mathbf{X}_s) ds + \int_0^t e^{-\alpha s} \mathcal{L}W(\mathbf{X}_s) ds \\
&+ \int_0^t e^{-\alpha s} \nabla W(\mathbf{X}_s) \cdot \sigma(\mathbf{X}_s) d\mathbf{B}_s \\
&+ \int_0^t e^{-\alpha s} \frac{\partial}{\partial x_n} W(\mathbf{X}_s) (dA_s^{(1),c} - dA_s^{(2),c}) + \sum_{0 < s \leq t} e^{-\alpha s} \Delta W(\mathbf{X}_s).
\end{aligned}$$

Using the following identity

$$W(\mathbf{x}) + \sum_{0 < s \leq t} e^{-\alpha s} \Delta W(\mathbf{X}_s) = W(\mathbf{X}_{0-}) + \sum_{0 \leq s \leq t} e^{-\alpha s} \Delta W(\mathbf{X}_s),$$

and taking expectation of both sides of (4.9) with respect to $P_{\mathbf{x}}$ and let $t \rightarrow \infty$, we get the following:

$$(4.10) \quad W(\mathbf{x}) = E_{\mathbf{x}} \left(\int_0^\infty e^{-\alpha t} (\alpha - \mathcal{L}) W(\mathbf{X}_t) dt \right) \\ - E_{\mathbf{x}} \left(\int_0^\infty e^{-\alpha t} \frac{\partial}{\partial x_n} W(\mathbf{X}_t) (dA_t^{(1),c} - dA_t^{(2),c}) \right) \\ - E_{\mathbf{x}} \left(\sum_{0 \leq t < \infty} e^{-\alpha t} \Delta W(\mathbf{X}_t) \right).$$

Therefore

$$(4.11) \quad k_S(\mathbf{x}) - W(\mathbf{x}) \\ = E_{\mathbf{x}} \left(\int_0^\infty e^{-\alpha t} [h(\mathbf{X}_t) - (\alpha - \mathcal{L}) W(\mathbf{X}_t)] dt \right) \\ + E_{\mathbf{x}} \left(\int_0^\infty e^{-\alpha t} \left[f_1(\mathbf{X}_t) + \frac{\partial}{\partial x_n} W(\mathbf{X}_t) \right] dA_t^{(1),c} \right) \\ + E_{\mathbf{x}} \left(\int_0^\infty e^{-\alpha t} \left[f_2(\mathbf{X}_t) - \frac{\partial}{\partial x_n} W(\mathbf{X}_t) \right] dA_t^{(2),c} \right) \\ + E_{\mathbf{x}} \left(\sum_{0 \leq t < \infty} e^{-\alpha t} \Delta W(\mathbf{X}_t) \right) \\ + E_{\mathbf{x}} \left(\sum_{0 \leq t < \infty} e^{-\alpha t} \left(\int_{X_{nt-}}^{X_{nt-} + \Delta A_t^{(1)}} f_1(\mathbf{X}_t) dy + \int_{X_{nt-} - \Delta A_t^{(2)}}^{X_{nt-}} f_2(\mathbf{X}_t) dy \right) \right).$$

By Theorem 4.1, the first three integrands in (4.11) are all nonnegative for the process \mathbf{X}_t staying in the region S .

Define the sets

$$\Gamma_+ = \{t \geq 0 : \Delta A_t^{(1)} > 0\}, \quad \Gamma_- = \{t \geq 0 : \Delta A_t^{(2)} > 0\},$$

then $\Gamma_+ \cap \Gamma_- = \emptyset$. Rewrite the last two expectations of (4.11) as

$$E_{\mathbf{x}} \left(\sum_{t \in \Gamma_+} e^{-\alpha t} \int_{X_{nt-}}^{X_{nt-} + \Delta A_t^{(1)}} \left[\frac{\partial}{\partial x_n} W(\mathbf{X}_t) + f_1(\mathbf{X}_t) \right] dy \right) \\ + E_{\mathbf{x}} \left(\sum_{t \in \Gamma_-} e^{-\alpha t} \int_{X_{nt-} - \Delta A_t^{(2)}}^{X_{nt-}} \left[-\frac{\partial}{\partial x_n} W(\mathbf{X}_t) + f_2(\mathbf{X}_t) \right] dy \right).$$

By Theorem 4.1 this quantity is nonnegative, and this shows $k_S(\mathbf{x}) \geq W(\mathbf{x})$, $\forall \mathbf{x} \in S$.

Due to the extension (4.6), we proved $k_S(\mathbf{x}) \geq W(\mathbf{x})$, $\forall \mathbf{x} \in \mathbb{R}^n$.

2. If $S = \mathbb{R}^{n-1} \times [a, b]$ and the process \mathbf{X}_t is the reflecting diffusion on S , then by Theorem 4.1, the first integral in (4.11) is obviously zero. As to the second and third integrals in (4.11), because $dA_t^{(1)}, dA_t^{(2)}$ are zero whenever \mathbf{X}_t is in $\mathbb{R}^{n-1} \times (a, b)$, while at the boundary where $A_t^{(1)}, A_t^{(2)}$ increases, the integrands

are zero, these two integrals are zero too. The last two expectations are also zero due to this construction hence $W(\mathbf{x}) = k_S(\mathbf{x}), \forall \mathbf{x} \in S$.

On the other hand, suppose $W(\mathbf{x}) = k_S(\mathbf{x}), \forall \mathbf{x} \in S$, then all the expectations in (4.11) must be zero. Assume $S = \mathbb{R}^{n-1} \times [\beta, \gamma]$ and at least one of the inequalities is true: $\beta(\bar{\mathbf{x}}) \neq g_1(\bar{\mathbf{x}}), \gamma(\bar{\mathbf{x}}) \neq g_2(\bar{\mathbf{x}})$, then due to the continuity of these four functions we know that the sum of the first three integrals in (4.11) is positive by Theorem 4.1. And because the sum of the last two expectations in (4.11) is nonnegative, it can be seen that $W(\mathbf{x}) < k_S(\mathbf{x})$. Therefore in order to have $W(\mathbf{x}) = k_S(\mathbf{x})$, S must be the region $\mathbb{R}^{n-1} \times [a, b]$.

Again by Theorem 4.1, we see that the processes \mathbf{X}_t and $A_t^{(i)}$ ($i = 1, 2$) must all be continuous in order to eliminate the last two expectations in (4.11), which implies $A_t^{(i)} = A_t^{(i)c}$ ($i = 1, 2$) when $\beta(\bar{\mathbf{x}}) = a(\bar{\mathbf{x}}), \gamma(\bar{\mathbf{x}}) = b(\bar{\mathbf{x}})$. Therefore $(\mathbf{X}_t, A_t^{(1)}, A_t^{(2)})$ must be the reflecting diffusion on $\mathbb{R}^{n-1} \times [a, b]$.

□

REMARK 4.4. *The possible jumps, $\Delta A_t^{(i)}, i = 1, 2$, only happen at time zero. When the process \mathbf{X}_t starts at a point outside the region $\mathbb{R}^{n-1} \times [a, b]$, the control brings it back to this region immediately, and after that, the process will be a continuous reflected diffusion.*

If we let $\gamma = (0, 0, \dots, 0, 1)^T$, then the reflected diffusion can be written as

$$(4.12) \quad d\mathbf{X}_t = \mu(\mathbf{X}_t)dt + \sigma(\mathbf{X}_t)dB_t + \gamma dA_t^{(1)} - \gamma dA_t^{(2)}, \quad t > 0,$$

where $A_t^{(1)}$ increases only at the boundary a and $A_t^{(2)}$ increases only at the boundary b .

We notice that the reflection only happens to the last component of the process. Since the two curves a and b are smooth and uniformly Lipschitz, if we let $n(\mathbf{x})$ be the inward normal for \mathbf{x} at the boundary, then we can show that there exist positive constants ν_1, ν_2 such that

$$\begin{aligned} \forall \mathbf{x} = (\bar{\mathbf{x}}, a(\bar{\mathbf{x}})), \quad (\gamma, n(\mathbf{x})) &\geq \nu_1, \\ \forall \mathbf{x} = (\bar{\mathbf{x}}, b(\bar{\mathbf{x}})), \quad (\gamma, n(\mathbf{x})) &\leq -\nu_2. \end{aligned}$$

By a localization technique and Theorem 4.3 in [13], it can be shown that there exists a unique solution $(\mathbf{X}_t, A_t^{(1)}, A_t^{(2)})$ to the reflected diffusion (4.12). This problem is called the Skorohod problem.

COROLLARY 4.5. *Under the Assumptions 3.2, 3.3 and 3.4 for f_1, f_2 and the definition (4.1) for function h , the solution $W \in C^{2, \dots, 2}(\mathbb{R}^n)$ and the functions $a(\bar{\mathbf{x}}), b(\bar{\mathbf{x}})$ in Theorem 4.1 are uniquely determined. The function $W(\mathbf{x})$ ($\mathbf{x} \in \mathbb{R}^n$) coincides with the optimal return function $W^*(\mathbf{x})$ given in (4.8).*

Proof. In the proof of Theorem 4.1 we have seen that the conditions given in this Theorem are necessary conditions for the value function of the stochastic control problem (4.8), and this value function as well as the functions $a(\bar{\mathbf{x}}), b(\bar{\mathbf{x}})$ are uniquely determined. □

Concluding Remarks. In this paper, we studied a multi-dimensional stochastic singular control problem via Dynkin game and Dirichlet form. The value function of the Dynkin game satisfies a variational inequality problem, where we showed the existence of the solution, and the integrated form of this value function turns out to be the value function of the singular control problem. By characterizing the regularities of the value function of the Dynkin game and its integrated version, we showed the

existence of a classical solution to the Hamilton-Jacobi-Bellman equation associated with this multi-dimensional singular control problem, and this kind of problems were traditionally solved through viscosity solutions. We also proved that the optimal control policy is given by two curves and the controlled process is the reflected diffusion between these two curves.

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